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Prescribed matchings extend to Hamiltonian cycles in hypercubes with faulty edges☆

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ABSTRACT

Ruskey and Savage asked the following question: for $n \geq 2$, does every matching in Q_n extend to a Hamiltonian cycle in Q_n? Fink showed that the answer is yes for every perfect matching, thereby proving Kreweras' conjecture. In this paper we consider the question in hypercubes with faulty edges. We show for $n \geq 2$ that every matching M of at most $2n-1$ edges extends to a Hamiltonian cycle in Q_n. Moreover, we prove that when $n \geq 4$ and M is nonempty this conclusion still holds even if Q_n has at most $n-1-\lceil \frac{|M|}{2} \rceil$ faulty edges, with one exception.

1. Introduction

The n-dimensional hypercube Q_n is one of the most popular and efficient interconnection networks. There is a large amount of literature on graph-theoretic properties of hypercubes as well as on their applications in parallel computing; see [9,12].

It is well known that Q_n is Hamiltonian for every $n \geq 2$. This statement dates back to 1872 [8]. Since then, research on Hamiltonian cycles in hypercubes satisfying certain additional properties has received considerable attention. Applications in parallel computing inspired the study of Hamiltonian cycles in hypercubes with faulty edges [2,16,19]. Dvořák [4] showed for $n \geq 2$ that every set of at most $2n-3$ edges of Q_n forming vertex-disjoint paths is contained in a Hamiltonian cycle. Wang and Chen [19] proved that this result still holds even if Q_n has some faulty edges; see Lemma 2.8. For more details about this topic, see [3,5,15].

Ruskey and Savage [14] asked the following question: for $n \geq 2$, does every matching in Q_n extend to a Hamiltonian cycle in Q_n? Kreweras [11] conjectured for $n \geq 2$ that every perfect matching of Q_n extends to a Hamiltonian cycle in Q_n. Fink [6,7] solved this conjecture by proving a stronger result. Also, Fink [6] pointed out that the statement is true for $n \in \{2, 3, 4\}$. The result in [4] implies that every matching of at most $2n-3$ edges extends to a Hamiltonian cycle in Q_n.

In a bipartite graph G, a set $S \subseteq V(G)$ is *deficient* if $|N(S)| < |S|$. A matching M with vertex set U is k-*suitable* if $G-U$ has no deficient set of size less than k. Vandenbussche and West [18] showed that if M is an induced matching or M is a matching having at most $k(n-k)+\frac{(k-1)(k-2)}{2}$ edges, where $1 \leq k \leq n-3$, then M extends to a perfect matching of Q_n and hence also extends to a Hamiltonian cycle in Q_n.

In this paper, we consider the question of Ruskey and Savage in faulty hypercubes and obtain the following main results: every matching M having at most $2n-1$ edges extends to a Hamiltonian cycle in Q_n when $n \geq 2$. Furthermore, when $n \geq 4$

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and M is nonempty the conclusion still holds even if at most $n-1-\lceil \frac{|M|}{2} \rceil$ edges are faulty (forbidden), with one exception. The rest of the paper is organized as follows. In Section 2 we introduce some necessary definitions and preliminaries. In Sections 3 and 4 we discuss the induction bases of the two main theorems. The main results are stated and proved in Section 5.

2. Definitions and preliminaries

Terminology and notation used in this paper but undefined below can be found in [1]. As usual, the vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$. For a set $F \subseteq E(G)$, let $G - F$ denote the graph with the vertex set $V(G)$ and edge set $E(G) \setminus F$. The *distance* between two vertices u and v is the number of edges in a shortest path joining u and v in G, denoted by $d_G(u, v)$, with the subscripts being omitted when the context is clear. For any two edges uv and xy, let $d(uv, xy) = \min\{d(u, x), d(u, y), d(v, x), d(v, y)\}$. Throughout the paper, n always denotes a positive integer, and $[n]$ denotes the set $\{1, \ldots, n\}$.

The *n-dimensional hypercube* Q_n is a graph whose vertex set consists of all binary strings of length n, with two vertices being adjacent whenever the corresponding strings differ in just one position. An edge in Q_n is an *i-edge* if its endpoints differ in the ith position. The set of all i-edges of Q_n is denoted by E_i. For $j \in [n]$, let Q^0_{n-1} and Q^1_{n-1} be the two $(n-1)$-dimensional subcubes of Q_n induced by all the vertices with the jth positions being 0 and 1, respectively. Since $Q_n - E_j = Q^0_{n-1} \cup Q^1_{n-1}$, we say that Q_n *splits* into two $(n-1)$-dimensional subcubes Q^0_{n-1} and Q^1_{n-1} by E_j.

Lemma 2.1 ([17]). *For $n \geq 2$, if e and f are two disjoint edges in Q_n, then Q_n splits into two $(n-1)$-dimensional subcubes such that one contains e and the other contains f.*

Note that Lemma 2.1 is just an observation, since e and f can exclude only two directions, and they can exclude two only if $n \geq 3$.

A *u,v-path* is a path with endpoints u and v, denoted by $P_{u,v}$ when we specify a particular such path. We say that a spanning subgraph of G whose components are k disjoint paths is a *spanning k-path* of G. A spanning 1-path thus is simply a spanning or Hamiltonian path. For a set $E' \subseteq E(G)$, a subgraph H of G *passes through* E' if $E' \subseteq E(H)$.

Let us recall the following classical result obtained by Havel [10] and several more recent results.

Lemma 2.2 ([10]). *If $n \geq 1$ and $u, v \in V(Q_n)$ are such that $d(u, v)$ is odd, then Q_n contains a spanning u, v-path.*

Lemma 2.3 ([4]). *For $n \geq 2$, if $u, v \in V(Q_n)$ and $e \in E(Q_n)$ are such that $d(u, v)$ is odd and $e \neq uv$, then Q_n contains a spanning u, v-path passing through edge e.*

Lemma 2.4 ([3]). *For $n \geq 3$, if $u, v \in V(Q_n)$ and $e \in E(Q_n)$ are such that $d(u, v)$ is odd, then Q_n contains a spanning u, v-path avoiding e.*

Let H_1 and H_2 be two subgraphs of G with $V(H_1) \cap V(H_2) = \emptyset$. We use $H_1 + H_2$ to denote the graph with the vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$.

Lemma 2.5 ([4]). *If $n \geq 2$ and x, y, u, v are distinct vertices in Q_n such that both $d(x, y)$ and $d(u, v)$ are odd, then Q_n contains a spanning 2-path $P_{x,y} + P_{u,v}$. Moreover, in the case when $d(x, y) = 1$, path $P_{x,y}$ can be chosen to have length 1, unless $n = 3$, $d(u, v) = 1$, and $d(xy, uv) = 2$.*

Note that for any edge $e \in E(Q_3)$ there exists a unique edge $f \in E(Q_3)$ such that $d(e, f) = 2$.

A forest is *linear* if each component of it is a path.

Lemma 2.6 ([4]). *For $n \geq 2$, if $E' \subseteq E(Q_n)$ are such that $|E'| \leq 2n - 3$, then there exists a Hamiltonian cycle containing E' in Q_n if and only if the subgraph with edge set E' is a linear forest.*

Lemma 2.7 ([13]). *If $n \geq 3$ and $F \subseteq E(Q_n)$ are such that $|F| \leq n - 2$, then any edge of $E(Q_n) \setminus F$ lies on some Hamiltonian cycle in $Q_n - F$.*

Lemma 2.8 ([19]). *For $n \geq 2$, let $F \subseteq E(Q_n)$ and $E' \subseteq E(Q_n) \setminus F$ with $1 \leq |E'| \leq 2n - 3$, and $|F| \leq n - 2 - \lfloor \frac{|E'|}{2} \rfloor$. If the subgraph with edge set E' is a linear forest, then all edges of E' lie on some Hamiltonian cycle in $Q_n - F$.*

3. Base step of induction for the first main theorem

Let K_{Q_n} denote the complete graph on the vertex set of the hypercube Q_n.