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A dense set of chromatic roots which is closed under multiplication by positive integers

Adam Bohn

Mathematics Research Institute, University of Exeter, Exeter, EX4 4QF, UK

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ABSTRACT

We study a very large family of graphs, the members of which comprise disjoint paths of cliques with extremal cliques identified. This broad characterisation naturally generalises those of various smaller families of graphs having well-known chromatic polynomials. We derive a relatively simple formula for an arbitrary member of the subfamily consisting of those graphs whose constituent clique-paths have at least one trivial extremal clique, and use this formula to show that the set of all non-integer chromatic roots of these graphs is closed under multiplication by natural numbers. A well-known result of Sokal then leads to our main result, which is that there exists a set of chromatic roots which is closed under positive integer multiplication in addition to being dense in the complex plane. Our findings lend considerable weight to a conjecture of Cameron, who has suggested that this closure property may be a generic feature of the chromatic polynomial. We also hope that the formula we provide will be of use to those computing with chromatic polynomials.

1. Introduction

Given some $q \in \mathbb{N}$ and a graph G (all graphs will be assumed to be simple, that is without loops or multiple edges), a *proper q-colouring* of G is a function from the vertices of G to a set of q colours, with the property that adjacent vertices receive different colours. The *chromatic polynomial* $P_G(X)$ of G is the unique monic polynomial which, when evaluated at q, gives the number of proper q-colourings of G. The chromatic polynomial has been the subject of a huge amount of research since it was introduced over a century ago by Birkhoff [3]. For a comprehensive introduction to this polynomial see [13].

A *chromatic root* of a graph is simply a zero of that graph's chromatic polynomial. Often we only aim to know whether or not a given number α is a zero of *some* chromatic polynomial, in which case we simply say that α is (not) a chromatic root, without specifying a graph.

Although a number of papers have been published on linear factors of chromatic polynomials [9,10], the bulk of the literature focuses on non-integral chromatic roots. These are somewhat mysterious, in that they are produced by a simple counting process and yet have no obvious combinatorial interpretation. With certain notable exceptions, such as Sokal's discovery [15] that the maximum degree of a graph determines a bound on the absolute values of its chromatic roots, little progress has been made towards an understanding of the mechanism relating a graph's abstract structure to properties of its non-integer chromatic roots.

On the other hand, the application of analytic methods to this subject has proved resoundingly successful in determining the distribution of chromatic roots on the real line and in the complex plane. The most important results in this area can be summarised by the following theorem, the three parts of which are due, respectively, to Jackson [12], Thomassen [17] and Sokal [16]:

E-mail address: a.s.bohn@exeter.ac.uk.

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Theorem 1. *The zero-free intervals and regions for the chromatic polynomial are entirely classified by the following* 3 *results:*

(i) *There are no negative real chromatic roots, and none in either of the intervals* $(0, 1)$ *or* $(1, 32/27]$.
(ii) *Chromatic roots are dense in the remainder of the real line.*
(iii) *Chromatic roots are dense in the whole complex plane.*

These results collectively represent quite a triumph in the study of chromatic polynomials: the results of Jackson and Thomassen provide a complete classification of those intervals of the real line in which zeros of chromatic polynomials do not occur, and Sokal's theorem – which the present work builds upon – states that there are no analogous "forbidden regions" for complex chromatic roots.[1] However, they are of little help when it comes to determining whether or not a given number having no conjugates in forbidden intervals is a chromatic root of some graph. As chromatic polynomials are monic with integer coefficients, their zeros are by definition algebraic integers; our current lack of insight into the converse question of which algebraic integers are chromatic roots is largely what motivates the current work.

The main result of this paper is the following extension of Sokal's result, which we shall prove by explicit construction.

Theorem 2. *There exists a set of chromatic roots which is dense in the complex plane, and closed under multiplication by natural numbers.*

It would be dubious to claim that this constitutes a strengthening of the previously stated analytic results. After all, we know that the set of all algebraic integers is dense in $\mathbb{C}$, and that this set strictly contains the set of all chromatic roots; as density of sets is not a property which can be qualified, we cannot hope to "improve" on Sokal's theorem. However, we are still very much lacking in knowledge of the algebraic properties of the chromatic polynomial,[2] and Theorem 2 does contribute in this respect, as we shall explain.

At the time of writing only a handful of algebraic integers outside of the forbidden intervals have been definitively shown not to be chromatic roots (notable examples are certain Beraha constants – numbers of the form $4\cos^2(\pi/n)$, $n \in \mathbb{N}$ – and some generalised Fibonacci numbers). To our knowledge not a single such number with non-zero imaginary part is known. In the absence of any reliable means via which to rule out a number being a chromatic root, two conjectures were proposed at a Newton Institute workshop in 2008 which address this issue from a quite different perspective (these appear in an as-yet-unpublished manuscript of Cameron and Morgan [7]). The first of these, known as the "$\alpha + n$ conjecture", essentially claims that chromatic polynomials are as "algebraically diverse" as can be, in that every number field is contained in the splitting field of some chromatic polynomial. This has been proved for quadratic fields, initially by participants of the aforementioned workshop. Later a different technique was used and extended to cubic fields by the present author; in [4] an algorithm is given to construct infinite families of bipartite graphs having complements with the desired chromatic splitting field.

The second proposed conjecture (the "$n\alpha$ conjecture"), which is strengthened by the work presented here, asserts that the set of all chromatic roots is closed under multiplication by positive integers. Notwithstanding the results of this paper, empirical evidence would appear to be in favour of the general conjecture. In particular a direct additive analogue follows from the well-known fact that if α is a chromatic root of a graph G, then $\alpha + n$ is a chromatic root of the join of G with K_n (this can be shown by a simple counting argument). However, it is difficult to see a way to approach a full proof of this conjecture. Theorem 2 is, to the best of our knowledge, the only real evidence to be found since the conjecture was proposed, and while it is certainly noteworthy, it does not entirely succeed in escaping the imprecision inherent in the analytical approach, a leap which will clearly need to be made if we are to further our algebraic understanding of the chromatic polynomial.

The remainder of this paper shall be dedicated to a proof of Theorem 2. In order to achieve this we will derive a general formula for a large class of graphs which generalise those used by Sokal to prove the density of chromatic roots in the complex plane.

2. Clique–theta graphs

We define a *generalised theta graph* $\Theta_{m_1,\ldots,m_n}$ to be a graph consisting of two vertices joined by n otherwise disjoint paths of lengths $m_1, \ldots, m_n$. In the study of the location of chromatic roots, generalised theta graphs have been the subject of some interest (see for example [6,5,11]). Most significantly, these were the graphs that Sokal used in his proof of the density result stated previously.[3]

With this in mind, it simplifies matters to approach our main result via the following more specialised proposition, which is essentially the $n\alpha$ conjecture restricted to generalised theta graphs.

[1] It should be stressed that Sokal's findings do not preclude the existence of curves which contain no chromatic roots; in fact, it is widely suspected that there are no purely imaginary chromatic roots.

[2] On the other hand, algebraic *methods*, such as that expounded by Biggs et al. [2,1], have been used to great effect in the study of the chromatic polynomial.

[3] In fact he was able to prove this result using the considerably smaller subfamily whose members have paths of uniform length. Note that this restriction allows only for specification of two variables: the number of paths, and length of the paths in a given graph. It seems quite surprising that a polynomial furnishing a dense set of zeros upon specialisation into $\mathbb{N}$ of just two parameters could occur so naturally.