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# On the order of graphs with a given girth pair

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ABSTRACT

A $(k; g, h)$-graph is a $k$-regular graph of girth pair $(g, h)$ where $g$ is the girth of the graph, $h$ is the length of a smallest cycle of different parity than $g$ and $g < h$. A $(k; g, h)$-cage is a $(k; g, h)$-graph with the least possible number of vertices denoted by $n(k; g, h)$. In this paper we give a lower bound on $n(k; g, h)$ and as a consequence we establish that every $(k; 6)$-cage is bipartite if it is free of odd cycles of length at most $2k-1$. This is a contribution to the conjecture claiming that every $(k; g)$-cage with even girth $g$ is bipartite. We also obtain upper bounds on the order of $(k; g, h)$-graphs with $g = 6, 8, 12$. From the proofs of these upper bounds we obtain a construction of an infinite family of small $(k; g, h)$-graphs. In particular, the $(3; 6, h)$-graphs obtained for $h = 7, 9, 11$ are minimal.



## 1. Introduction

A $(k; g)$-*cage* is a $k$-regular graph of girth $g$ having the minimum possible number of vertices which is denoted by $n(k; g)$. Harary and Kóvacs [12] generalize the concept of $(k; g)$-cages by replacing the girth with a *girth pair condition* $(g, h)$, (i.e., $g$ is the girth of the graph, $h$ is the length of a smallest cycle of different parity than $g$ and $g < h$). The authors of [12] proved the existence of $(k; g, h)$-cages with $3 \leq g < h$, obtaining that their order $n(k; g, h)$ must fulfill the inequality $n(k; g, h) \leq 2n(k; h)$. Also, they proved that if $k \geq 3$ and $h \geq 4$, then $n(k; h-1, h) \leq n(k; h)$. In [17] the strict inequality $n(k; h-1, h) < n(k; h)$ for $k \geq 3$ and $h \geq 4$ is proved. The exact values $n(k; 4, h)$ are studied in [13,15,18] and the exact values $n(3; 6, 7) = 18$, $n(3; 6, 9) = 24$ and $n(3; 6, 11) = 28$ are determined in [7]. Moreover [4] contains a lower bound on $n(k; g, h)$ for odd $g \geq 5$ and even $h > g$.

In this paper we obtain a lower bound on the order of a $(k; g, h)$-graph with $g \geq 6$ even and $h \geq g+1$ odd. Let $n_0(k; g)$ denote the lower bound on the order of a $k$-regular graph with girth $g$. Biggs and Ito [6] proved that every $k$-regular graph with even girth $g \geq 6$ and order at most $n_0(k; g)+k-2$ must be bipartite. As a consequence of our lower bound we improve this result for $g = 6$ proving that every $k$-regular graph with $k \geq 3$, girth 6 and order at most $n_0(k; 6) + 2k^2 - 6k + 1$ free of odd cycles of length at most $2k-1$ must be bipartite. Furthermore, it is conjectured that cages with even girth are bipartite [14,16]. Applying our results we also contribute to this conjecture establishing that every $(k; 6)$-cage is bipartite provided that it is free of odd cycles of length at most $2k-1$. We also obtain an upper bound on the order of a $(k; g, h)$-graph with $g = 6, 8, 12$ even and $h \geq g+1$ odd. From the proofs of these upper bounds we obtain a construction of small $(k; g, h)$-graphs using two copies of a $(k; g)$-cage. In particular the $(3; 6, h)$-graphs obtained are minimal.

## 2. Terminology and known results

All graphs considered are finite, undirected and simple (without loops or multiple edges). For definitions and notations not explicitly stated the reader may refer to [8].

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Let $G$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $U \subset V$ the subgraph induced by $U$ is denoted by $G[U]$. A path between a vertex $u$ and a vertex $v$ will be called a $uv$-path. The distance $d_G(u, v)$ between two vertices $u$ and $v$ is the minimum of the lengths among the $uv$-paths of $G$. The *girth* of a graph $G$ is the length $g = g(G)$ of a shortest cycle. A *girdle* is a shortest cycle. The *neighborhood* $N(u) = N_G(u)$ of a vertex $u$ is the set of its neighbors i.e., vertices adjacent to $u$. The *closed neighborhood* of $u$ is $N[u] = N(u) \cup \{u\}$ and the neighborhood of a subset $U \subset V$ is defined as $N(U) = \cup_{u \in U} N(u)$. The *degree* of a vertex $v \in V$ is the cardinality of $N(u)$. A graph is called *k-regular* if all its vertices have the same degree $k$. A $(k; g)$-*graph* is a $k$-regular graph of girth $g$ and a $(k; g)$-*cage* is a $(k; g)$-graph with the smallest possible number of vertices. The existence of a $(k; g)$-cage was established by Erdős and Sachs [9]. For $k \geq 3$ and $g \geq 5$ the order $n(k; g)$ of a cage is bounded by

$$n_0(k; g) = \begin{cases} 1 + k \displaystyle\sum_{i=0}^{(g-3)/2} (k-1)^i & g \text{ odd;} \\ 2 \displaystyle\sum_{i=0}^{(g-2)/2} (k-1)^i & g \text{ even.} \end{cases} \qquad (1)$$

This bound is known as the *Moore bound* for cages and cages attaining the Moore bound are called *Moore cages*. Moore cages of even girth exist only for $g \in \{4, 6, 8, 12\}$ [5]. For $g = 4$, they are the complete $k$-regular bipartite graphs. For $g = 6, 8, 12$, these graphs are constructed as the incidence graphs of certain finite geometries whenever $k - 1$ is a prime power. More details about constructions of cages can be found in the survey by Wong [16] or in the dynamic cage survey by Exoo and Jajcay [10].

The paper is organized as follows. In the following section we establish some lower bounds on $n(k; g, h)$ for $g \geq 6$ even and $h \geq g+1$ odd. As a consequence we prove that every $(k; 6)$-graph with $k \geq 3$ and order at most $n_0(k; 6)+2k^2-6k+1 = 4k^2 - 4k + 3$ free of odd cycles of length at most $2k - 1$ must be bipartite. Moreover, we show that every $(k; 6)$-cage is bipartite if it is free of odd cycles of length at most $2k - 1$. In the final section we establish some upper bounds on $n(k; g, h)$ for $g = 6, 8, 12$ and $h > g$ odd. From the proofs of these upper bounds we obtain a construction of small $(k; g, h)$-graphs using two copies of a $(k; g)$-cage. In particular for $q = 2$ we have a construction of $(3; 6, h)$-cages for $h = 7, 9, 11$, having $n(3; 6, 7) = 18$, $n(3; 6, 9) = 24$ and $n(3; 6, 11) = 28$ vertices respectively. These exact values were already proved in [7] and we have checked that each of our graphs is isomorphic to the graphs previously obtained in [7].

## 3. Bounds

### *3.1. Lower bounds*

Biggs and Ito [6] proved that every $(k; g)$-graph with even girth $g \geq 6$ and order at most $n_0(k; g)+k-2$ must be bipartite. As an immediate consequence of this result we can write the following lower bound.

**Corollary 3.1.** *$n(k; g, h) \geq n_0(k; g) + k - 1$ for $k \geq 3$, $g \geq 6$ even and $h$ odd.*

By (1) we have $n_0(k; 6) = 2(k^2 - k + 1)$. Then, for $k = 3$ and $g = 6$, Corollary 3.1 yields $n(3; 6, h) \geq 16$. We find the following result in [7] which is an improvement of Corollary 3.1 for $k = 3$ and $g = 6$.

**Theorem 3.1** ([7]). *$n(3; 6, h) \geq (7h + 1)/3$ for $h$ odd.*

Our first objective is to improve Corollary 3.1 for $g = 6$ extending Theorem 3.1 for any degree $k \geq 3$. We need to prove two lemmas.

**Lemma 3.1.** *Let $G$ be a $(k; g, h)$-graph with $k \geq 3$, $g \geq 6$ even and $h \geq g + 1$ odd. Let $\gamma$ be an $h$-cycle of $G$. Then every vertex of $G - V(\gamma)$ is adjacent to at most one vertex of $\gamma$.*

**Proof.** Note that $\gamma$ is an induced subgraph of $G$ since $\gamma$ has no chord, otherwise an odd $h'$-cycle with $h' < h$ results in $G$ which is a contradiction. If some vertex $z$ of $G - V(\gamma)$ is adjacent to $u, v \in V(\gamma)$ and $d_\gamma(u, v) = \ell$, then $G$ contains two cycles, one of length $\ell + 2$ and another of length $h - \ell + 2$. If $\ell$ is even, $\ell + 2 \geq g$ and $h - \ell + 2 \geq h$ must hold. Consequently, $\ell \leq 2$, implying that $\ell + 2 \leq 4$ which is a contradiction because $\ell + 2 \geq g \geq 6$. Therefore $\ell$ is odd, $\ell + 2 \geq h$ and $h - \ell + 2 \geq g$ must hold. Then, from these two inequalities we obtain $h - \ell + 2 \leq h - (h - 2) + 2 = 4$ which is again a contradiction. ■

**Lemma 3.2.** *Let $G$ be a $(k; g, h)$-graph with $k \geq 3$, $g \geq 6$ even and $h \geq g + 1$ odd. Let $\gamma$ be an $h$-cycle of $G$ and $w$ any vertex in $N(\gamma) \setminus V(\gamma)$. If $g = 6$, $w$ is adjacent to at most one vertex in $N(\gamma) \setminus V(\gamma)$; and if $g \geq 8$, $w$ is adjacent to no vertex in $N(\gamma) \setminus V(\gamma)$.*

**Proof.** We reason by contradiction assuming that there are $x, y, z \in N(\gamma) \setminus V(\gamma)$ such that $x, z \in N(y)$. Let $u_x, u_y, u_z \in V(\gamma)$ be such that $u_x x, u_y y, u_z z \in E(G)$. Let $\ell_1, \ell_2$ are defined as the lengths of the $(u_x, u_y)$-path and the $(u_y, u_z)$-path in an orientation of $\gamma$, where the three vertices appear in the order $(u_x, u_y, u_z)$. Then the length of the $(u_x, u_z)$-path is $\ell_1 + \ell_2$