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Inversion polynomials for 321-avoiding permutations

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ABSTRACT

We prove a generalization of a conjecture of Dokos, Dwyer, Johnson, Sagan, and Selsor giving a recursion for the inversion polynomial of 321-avoiding permutations. We also answer a question they posed about finding a recursive formula for the major index polynomial of 321-avoiding permutations. Other properties of these polynomials are investigated as well. Our tools include Dyck and 2-Motzkin paths, polyominoes, and continued fractions.

1. Introduction

The main motivation for this paper is a conjecture of Dokos, Dwyer, Johnson, Sagan, and Selsor [8] about inversion polynomials for 321-avoiding permutations which we will prove in generalized form. We also answer a question they posed by giving a recursive formula for the analogous major index polynomials. We first introduce some basic definitions and notation about pattern avoidance and permutation statistics.

Call two sequences of distinct integers $\pi = a_1 \ldots a_k$ and $\sigma = b_1 \ldots b_k$ *order isomorphic* whenever $a_i < a_j$ if and only if $b_i < b_j$ for all i, j. Let $\mathfrak{S}_n$ denote the symmetric group of permutations of $[n] \stackrel{\text{def}}{=} \{1, \ldots, n\}$. Say that $\sigma \in \mathfrak{S}_n$ *contains* $\pi \in \mathfrak{S}_k$ *as a pattern* if there is a subsequence σ' of σ order isomorphic to π. If σ contains no such subsequence then we say σ *avoids* π and write $\mathrm{Av}_n(\pi)$ for the set of such $\sigma \in \mathfrak{S}_n$.

Let $\mathbb{Z}$ and $\mathbb{N}$ denote the integers and nonnegative integers, respectively. A *statistic* on $\mathfrak{S}_n$ is a function $\mathrm{st} : \mathfrak{S}_n \to \mathbb{N}$. One then has the corresponding generating function

$$f_n^{\mathrm{st}} = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{st}\,\sigma}.$$

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Two of the most ubiquitous statistics for $\sigma = b_1 \dots b_n$ are the *inversion number*

$$\operatorname{inv}\sigma = \#\{(i,j) \mid i < j \text{ and } b_i > b_j\}$$

where the hash sign denotes cardinality, and the *major index*

$$\operatorname{maj}\sigma = \sum_{b_i > b_{i+1}} i.$$

In [21], Sagan and Savage proposed combining the study of pattern avoidance and permutations statistics by considering generating functions of the form

$$F^{\mathrm{st}}(\pi) = \sum_{\sigma \in \mathrm{Av}_n(\pi)} q^{\mathrm{st}\,\sigma} \tag{1}$$

for any pattern π and statistic st. The earliest reference of which we are aware combining the inversion statistic and pattern avoidance is the paper of Barcucci et al. [2] where generating trees are used. Various subsequent articles also considered inversions in restricted permutations such as [1,3,5,9,18,20]. Dokos et al. [8] were the first to carry out an extensive study of the generating functions of the form (1) for the inv and maj statistics. We note that when st $=$ inv and $\pi = 132$ we recover a q-analogue of the Catalan numbers studied by Carlitz and Riordan [4]. Work on the statistics counting fixed points and excedances has been done by Elizalde [11,10], Elizalde and Deutsch [12], and Elizalde and Pak [13].

Our primary motivation was to prove a conjecture of Dokos et al. concerning the inversion polynomial for 321-avoiding permutations. In fact, we will prove a stronger version which also keeps track of left–right maxima. Call a_i in $\pi = a_1 \dots a_n$ a *left–right maximum (value)* if $a_i = \max\{a_1, \dots, a_i\}$. We let

$$\operatorname{Lrm}\pi = \{a_i \mid a_i \text{ is a left–right maximum}\}$$

and $\operatorname{lrm}\pi = \#\operatorname{Lrm}\pi$. Consider the generating function

$$I_n(q,t) = \sum_{\sigma \in \mathrm{Av}_n(321)} q^{\operatorname{inv}\sigma} t^{\operatorname{lrm}\sigma}. \tag{2}$$

Note that since $\#\mathrm{Av}_n(321) = C_n$, the nth Catalan number, this polynomial is a q, t-analogue of C_n. Our main result is a recursion for $I_n(q,t)$. The case $t = 1$ was a conjecture of Dokos et al.

Theorem 1.1. *For $n \geq 1$,*

$$I_n(q,t) = tI_{n-1}(q,t) + \sum_{k=0}^{n-2} q^{k+1} I_k(q,t) I_{n-k-1}(q,t).$$

We should note that after seeing a version of this article in preprint form, Mansour and Shattuck [19] have given a proof of the special case $t = 1$ of this recursion using formal manipulation of continued fractions. In fact, with this substitution, their main result is just our Corollary 7.5 below.

The rest of this paper is structured as follows. In the next section we will give a direct bijective proof of Theorem 1.1 using 2-Motzkin paths. The following two sections will explore related ideas involving Dyck paths, including a combinatorial proof of a formula of Fürlinger and Hofbauer [15] and two new statistics which are closely related to inv. Sections 5 and 6 are devoted to polyominoes. First, we give a second proof of Theorem 1.1 using work of Cheng, Eu, and Fu [7]. Next we derive recursions for a major index analogue, $M_n(q,t)$, of (2), thus answering a question posed by Dokos et al. in their paper. In the final section we use continued fractions to prove a refined version of Theorem 1.1 where we also keep track of the number of fixed points. There are a number of properties of $I_n(q,t)$ and $M_n(q,t)$ (symmetry, unimodality, modulo 2 behavior, and signed enumeration) which are tangentially related to the present work. So we have collected them in an addendum available on the arXiv [6].

2. A proof of Theorem 1.1 using 2-Motzkin paths

Our first proof of Theorem 1.1 will use 2-Motzkin paths. Let U (up), D (down), and L (level) denote vectors in $\mathbb{Z}^2$ with coordinates $(1,1)$, $(1,-1)$, and $(1,0)$, respectively. A *Motzkin path of length n*, $M = s_1 \dots s_n$, is a lattice path where each step s_i is U, D, or L and which begins at the origin, ends on the x-axis, and never goes below $y = 0$. A 2-*Motzkin path* is a Motzkin path where each level step has been colored in one of two colors which we will denote by L_0 and L_1. We will let $\mathcal{M}_n^{(2)}$ denote the set of 2-Motzkin paths of length n.

It will be useful to have two vectors to keep track of the values and positions of left–right maxima. If $\sigma = b_1 \dots b_n \in \mathfrak{S}_n$ then let

$$\operatorname{val}\sigma = (v_1, \dots, v_n)$$

where

$$v_i = \begin{cases} 1 & \text{if } i \text{ is a left–right maximum of } \sigma, \\ 0 & \text{else.} \end{cases}$$