Discrete Mathematics 313 (2013) 2626–2637

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

On globally sparse Ramsey graphs

Torsten Mütze, Ueli Peter *

Institute of Theoretical Computer Science, ETH Zürich, 8092 Zürich, Switzerland

ARTICLE INFO

Article history:
Received 1 June 2012
Received in revised form 17 July 2013
Accepted 30 July 2013
Available online 25 August 2013

Keywords:
Ramsey theory
Graph density
Arboricity
Nash-Williams theorem

ABSTRACT

We say that a graph G has the Ramsey property w.r.t. some graph F and some integer $r \geq 2$, or G is (F, r)-Ramsey for short, if any r-coloring of the edges of G contains a monochromatic copy of F. Rödl and Ruciński asked how globally sparse (F, r)-Ramsey graphs G can possibly be, where the density of G is measured by the subgraph $H \subseteq G$ with the highest average degree. So far, this so-called Ramsey density is known only for cliques and some trivial graphs F. In this work we determine the Ramsey density up to some small error terms for several cases when F is a complete bipartite graph, a cycle or a path, and $r \geq 2$ colors are available.

1. Introduction

Ramsey's celebrated theorem [22] states that for any integers r and ℓ, *any* r-coloring of the edges of a large enough complete graph contains a monochromatic clique on ℓ vertices, i.e., a clique whose edges all receive the same color. In this context we say that a graph G *has the Ramsey property w.r.t. some graph F and some integer* $r \geq 2$, or G is (F, r)-*Ramsey* for short, if any r-coloring of the edges of G contains a monochromatic copy of F. While Ramsey's theorem seems to rely on the fact that a large complete graph is very dense, Folkman [9] proved that there are graphs that are Ramsey with respect to K_ℓ and $r = 2$ colors which do not contain a $K_{\ell+1}$ as a subgraph. This result was later generalized by Nešetřil and Rödl [19] to the case of more than 2 colors. The smallest currently known graph that is $(K_3, 2)$-Ramsey and K_4-free has 941 vertices [7].

Not allowing a $K_{\ell+1}$-subgraph is an entirely *local* density restriction and still allows for graphs that are very dense *globally*, in the sense that they contain many edges. Motivated by this fact, Rödl and Ruciński [23] asked how globally sparse Ramsey graphs can possibly be. They introduced the *Ramsey density of F and r*, defined as

$$m^*(F, r) := \inf\{m(G) \mid G \text{ is } (F, r)\text{-Ramsey}\}, \tag{1}$$

where

$$m(G) := \max_{H \subseteq G} \frac{e(H)}{v(H)}, \tag{2}$$

and $e(H)$ and $v(H)$ denote the number of edges and vertices of H, respectively. The parameter $m(G)$ measures the global density of G; it is equal to half the average degree of H, maximized over all subgraphs $H \subseteq G$. This density parameter and variations of it arise naturally in the theory of random graphs [4,13], and also in Nash-Williams' theorem for the arboricity of a graph [20] (as we shall see this theorem actually plays a crucial role in our proofs).

Kurek and Ruciński [15] proved the somewhat surprising fact that the sparsest graph that is (K_ℓ, r)-Ramsey (in the sense of (1)) is a large complete graph on as many vertices as the Ramsey number $R(K_\ell, r)$ tells us; recall that the *Ramsey number*

* Corresponding author.
E-mail addresses: muetzet@inf.ethz.ch (T. Mütze), upeter@inf.ethz.ch (U. Peter).

0012-365X/$ – see front matter
http://dx.doi.org/10.1016/j.disc.2013.07.023

$R(F, r)$ *of* F *and* r is defined as the minimal $N = N(F, r)$ such that K_N is (F, r)-Ramsey. Their result shows that the Ramsey density of cliques is

$$m^*(K_\ell, r) = m(K_{R(K_\ell,r)}) = \frac{R(K_\ell, r) - 1}{2}. \quad (3)$$

Apart from cliques, the only graphs for which the Ramsey density is known exactly are the trivial cases of stars S_ℓ with ℓ rays and $r \geq 2$ colors and the path P_3 on 3 edges and $r = 2$ colors: For stars an easy pigeonholing argument shows that

$$m^*(S_\ell, r) = m(S_{r(\ell-1)+1}) = \frac{r(\ell - 1) + 1}{r(\ell - 1) + 2}. \quad (4)$$

For P_3 we have $m^*(P_3, 2) = 1$, which is also not hard to see.[1]

Also for an analogous parameter defined for *vertex*-colorings, the so-called *vertex-Ramsey density* introduced in [23] and further studied in [14], relatively little is known (even though one might suspect that vertex-colorings are much easier to deal with than edge-colorings). The authors of [14] offered a prize money of 400,000 złoty (Polish currency in 1993) for the exact determination of the vertex-Ramsey density for the case where the forbidden graph F is the path on 3 vertices and $r = 2$ colors are available.

1.1. Our results

In this work we determine the Ramsey density $m^*(F, r)$ up to some small error terms for several cases when F is a complete bipartite graph, a cycle or a path, and $r \geq 2$ colors are available.

Complete bipartite graphs. The first theorem summarizes our results for the case where F is a complete bipartite graph $K_{a,b}$, $a \leq b$. In [17] a general upper bound of $m^*(K_{a,b}, r) < r(a-1)+1$ has been derived. We are able to prove an almost matching lower bound for the case where b is somewhat larger than a.

Theorem 1 (*Complete bipartite graphs*)**.** *For any integers* $a \geq 2$, $b \geq (a-1)^2 + 1$ *and* $r \geq 2$ *we have*

$$r(a - 1) - \varepsilon \leq m^*(K_{a,b}, r) < r(a - 1) + 1, \quad (5)$$

where $\varepsilon = \varepsilon(a, b, r) := \frac{r(a-1)-1}{\max\{R(K_{a,b},r),2r(a-1)+1\}} < 1/2$.

From the best known general lower bound $R(K_{a,b}, r) \geq (2\pi\sqrt{ab})^{\frac{1}{a+b}} \left(\frac{a+b}{e^2}\right) r^{\frac{ab-1}{a+b}}$ from [5], it follows that ε tends to 0 for larger values of a, b and/or r. See [5,21] for better lower bounds on $R(K_{a,b}, r)$ in special cases that can be plugged into the lower bound in (5); see also the remarks at the end of this paper.

We note here that the upper bound for complete bipartite graphs $K_{a,b}$ stated in Theorem 1 (which holds for arbitrary values of a and b) can be slightly improved; see the remarks at the end of this paper.

Cycles. The next theorem summarizes our results for the case where F is a cycle C_ℓ. The upper bound for even cycles and the lower bound for odd cycles follow from results presented in [23]. For even cycles we are able to prove an almost matching lower bound.

Theorem 2 (*Cycles*)**.** *For any even integer* $\ell \geq 4$ *and any integer* $r \geq 2$ *we have*

$$r - \varepsilon \leq m^*(C_\ell, r) < r + 1, \quad (6)$$

where $\varepsilon = \varepsilon(\ell, r) := \frac{r-1}{\max\{R(C_\ell,r),2r+1\}} < 1/2$.

There is a function $f()$ *such that for any odd integer* $\ell \geq 3$ *and any integer* $r \geq 2$ *we have*

$$2^{r-1} \leq m^*(C_\ell, r) \leq f(r). \quad (7)$$

The dominant terms r and 2^r in these bounds for even and odd cycles, respectively, are very similar to those known for the Ramsey number $R(C_\ell, r)$ (see [3,11,18]). However, Theorem 2 shows that, unlike the Ramsey number, the Ramsey density does not grow unbounded for fixed r and $\ell \to \infty$.

Using the best known general lower bound $R(C_\ell, r) \geq (r-1)(\ell-2)+2$ from [25], it follows that ε tends to 0 for larger values of ℓ and/or r. See [21,25] for better lower bounds on $R(C_\ell, r)$ in special cases that can be plugged into the lower bound in (6); see also the remarks at the end of this paper.

---

1 For the lower bound proof note that any graph G with $m(G) < 1$ is a forest, and that each tree in this forest can be rooted and colored level by level with alternating colors, thus avoiding a monochromatic P_3. For the upper bound proof consider the 5-cycle with an additional dangling edge attached to every vertex.