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Relating the annihilation number and the 2-domination number of a tree

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ABSTRACT

A set S of vertices in a graph G is a 2-*dominating set* if every vertex of G not in S is adjacent to at least two vertices in S. The 2-*domination number* $\gamma_2(G)$ is the minimum cardinality of a 2-dominating set in G. The *annihilation number* $a(G)$ is the largest integer k such that the sum of the first k terms of the nondecreasing degree sequence of G is at most the number of edges in G. The conjecture-generating computer program, Graffiti.pc, conjectured that $\gamma_2(G) \le a(G) + 1$ holds for every connected graph G. It is known that this conjecture is true when the minimum degree is at least 3. The conjecture remains unresolved for minimum degree 1 or 2. In this paper, we prove that the conjecture is indeed true when G is a tree, and we characterize the trees that achieve equality in the bound. It is known that if T is a tree on n vertices with n_1 vertices of degree 1, then $\gamma_2(T) \le (n + n_1)/2$. As a consequence of our characterization, we also characterize trees T that achieve equality in this bound.

1. Introduction

In this paper, we study upper bounds on the 2-domination numbers of trees in terms of their annihilation numbers. For $k \ge 1$, a k-*dominating set* of a graph G is a set S of vertices in G such that every vertex outside S is adjacent to at least k vertices in S. Every graph G has a k-dominating set, since $V(G)$ is such a set. The k-*domination number* of G, denoted by $\gamma_k(G)$, is the minimum cardinality of a k-dominating set of G. In particular, a 1-dominating set is a dominating set, and the 1-domination number $\gamma_1(G)$ is the domination number $\gamma(G)$. A k-dominating set of G of cardinality $\gamma_k(G)$ is called a γ_k-*set* of G. The concept of a k-dominating set was first introduced by Fink and Jacobson in 1985 [6] and is now well-studied in the literature. We refer the reader to the two books on domination by Haynes, Hedetniemi, and Slater [9,10], as well as to the excellent survey on k-domination in graphs by Chellali, Favaron, Hansberg, and Volkmann [2].

As explained in [11], the *annihilation number* of a graph was first introduced by Pepper in [12]. Originally it was defined in terms of a reduction process on the degree sequence similar to the Havel–Hakimi process (see [7,13]). In [12], Pepper showed an equivalent way to define the annihilation number, this is the version we will use in this work. The *annihilation number* of a graph G, denoted $a(G)$, is the largest integer k such that the sum of the first k terms of the degree sequence of G arranged in nondecreasing order is at most the number of edges. That is if $d_1, \ldots, d_n$ is the degree sequence of a graph G with m edges, where $d_1 \le \cdots \le d_n$, then the annihilation number of G is the largest integer k such that $\sum_{i=1}^{k} d_i \le m$ or, equivalently, the largest integer k such that $\sum_{i=1}^{k} d_i \le \sum_{i=k+1}^{n} d_i$.

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The conjecture-generating computer program, Graffiti.pc, made the following conjecture relating the 2-domination number of a graph and its annihilation number.

Conjecture 1 ([3]). *If G is a connected graph with at least 2 vertices, then $\gamma_2(G) \leq a(G) + 1$.*

It is known that Conjecture 1 is true when the minimum degree is at least 3. Conjecture 1 is still unresolved when the minimum degree of G is 1 or 2. Proving the conjecture for trees may be the most interesting case. Our aim in this paper is threefold: first to prove that Conjecture 1 is indeed true for trees. Secondly to characterize the extremal trees achieving equality in the upper bound of Conjecture 1. Thirdly to characterize trees with the largest possible 2-domination number.

1.1. Notation

In this paper, the word "graph" is used to denote a "simple graph" with no loops or multiple edges. For notation and graph theory terminology not defined herein, we in general follow [9]. We write $V(G)$ and $E(G)$ for the vertex set and edge set of a graph G. Usually, we use n for the number of vertices and m for the number of edges. We write $N_G(v)$ and $d_G(v)$ for the neighborhood and degree of a vertex $v \in V(G)$. We extend the notion of neighborhood to sets by letting $N_G(S) = \bigcup_{v\in S} N(v)$ for any $S \subseteq V(G)$. If the graph G is clear from the context, we simply write $N(v)$, $N(S)$, and $d(v)$ rather than $N_G(v)$, $N_G(S)$, and $d_G(v)$, respectively. The minimum degree among the vertices of G is denoted by $\delta(G)$. The *matching number* is the maximum size of a matching in G and is denoted by $\alpha'(G)$. A vertex of degree 1 is called a *leaf*, its neighbor is a *support vertex*, and its incident edge is a *pendant edge*. We denote the set of leaves of a tree T by $L(T)$. A *star* is a tree with at most one non-leaf vertex. The *corona* of a graph G, denoted $G \circ K_1$, is formed from G by adding for each $v \in V(G)$, a new vertex v' and the pendant edge vv'.

For a set $S \subseteq V(G)$, we let $G[S]$ denote the subgraph induced by S. The graph obtained from G by deleting the vertices in S and all edges incident with vertices in S is denoted by $G - S$. In the special case when $S = \{v\}$, we also denote $G - S$ by $G - v$ for simplicity. For a set $S \subseteq V(G)$ and $v \in V(G)$, we denote by $d_S(v)$ the number of all vertices in S that are adjacent to v. In particular, when $S = V(G)$, we note $d_S(v) = d(v)$. For a subset $S \subseteq V(G)$, we define

$$\Sigma(S, G) = \sum_{v\in S} d_G(v).$$

For a graph G with m edges, we define an a-set of G to be a (not necessarily unique) set S of vertices in G such that $|S| = a(G)$ and $\sum_{v\in S} d_G(v) \leq m$. We define an $a_{\min}$-set of G to be an a-set S of G, such that $\Sigma(S, G)$ is a minimum. Hence if S is an $a_{\min}$-set of G, then S is a set of (not necessarily unique) vertices corresponding to the first $a(G)$ vertices in the nondecreasing degree sequence of G.

In order to prove Conjecture 1 for trees, we introduce a slight variation of the annihilation number of a graph. We define the *upper annihilation number* of a graph G, denoted $a^*(G)$, to be the largest integer k such that the sum of the first k terms of the degree sequence of G arranged in nondecreasing order is at most $|E(G)| + 1$. That is if $d_1, \ldots, d_n$ is the degree sequence of a graph G with m edges, where $d_1 \leq \cdots \leq d_n$, then the upper annihilation number of G is the largest integer k such that $\sum_{i=1}^{k} d_i \leq m + 1$. We define an $a^*_{\min}$-set of G to be a (not necessarily unique) set S^* of vertices in G such that $|S^*| = a^*(G)$ and S^* corresponds to the first $a^*(G)$ vertices in the nondecreasing degree sequence of G.

1.2. Known results and observations

In their introductory paper on k-domination, Fink and Jacobson [6] established the following lower bound on the k-domination number of a tree.

Theorem 1 ([6]). *For $k \geq 1$, if T is a tree with n vertices, then $\gamma_k(T) \geq ((k-1)n + 1)/k$.*

As a special case of Theorem 1, if T is a tree with n vertices, then $\gamma_2(T) \geq (n+1)/2$. The following upper bound on the 2-domination number of a tree was observed in several papers.

Theorem 2 ([5,8,14]). *If T is a tree with n vertices and n_1 leaves, then $\gamma_2(T) \leq (n + n_1)/2$.*

Caro and Roditty [1] and Strake and Volkmann [15] established the following upper bound on the k-domination number of a graph.

Theorem 3 ([1,15]). *For every graph G with n vertices and every integer $k \geq 1$, if $\delta(G) \geq 2k - 1$, then $\gamma_k(G) \leq \lfloor n/2 \rfloor$.*

In the special case when $k = 2$, the result of Theorem 3 states that if G is a graph with n vertices and $\delta(G) \geq 3$, then $\gamma_2(G) \leq \lfloor n/2 \rfloor$. Since $\alpha'(G) \leq \lfloor n/2 \rfloor$ for any graph G with n vertices, this result was improved in the following theorem.

Theorem 4 ([4]). *Let k be a positive integer. If G is any graph with $\delta(G) \geq 2k - 1$, then $\gamma_k(G) \leq \alpha'(G)$.*

We remark that both Theorems 2 and 3 follow from a more general result in Hansberg et al. [8].