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The critical group of a clique-inserted graph

Haiyan Chen [a,*], Fuji Zhang [b]

[a] *School of Sciences, Jimei University, Xiamen 361021, China*
[b] *Institute of Mathematics, Xiamen University, Xiamen 361005, China*

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ABSTRACT

In this paper, we consider the relations between the critical group of a regular graph G and that of its clique-inserted graph (or para-line graph) $C(G)$. First, we construct a group homomorphism between these two critical groups of G and $C(G)$. Based on the homomorphism, we show that the critical group of G is isomorphic to a quotient of that of $C(G)$ if G is not bipartite, and the minimal number of generators for the critical group of $C(G)$ is equal to the number of independent cycles in G if G is 2-edge connected. Second, by computing the Smith normal form of the Laplacian matrix of a graph, we obtain invariant factors of critical groups for some small regular graphs and their corresponding clique-inserted graphs.

1. Introduction

Let G be a connected graph. The *critical group* $K(G)$ of G is a finite abelian group whose order is the tree-number $\kappa(G)$ of G. The critical group $K(G)$ is a subtle isomorphism invariant of G, closely related to the cycle and bond spaces of the graph, the graph Laplacian [19,20], as well as the chip-firing game (or the abelian sandpile model) on the graph. Some of the standard results on $K(G)$ can be found in [1,5,6] and [12, Chapter 13]. But in general the relations between the critical group $K(G)$ and many other parameters of a graph G remain mysterious. So the critical group of a graph is an interesting topic for studying. Now the researches mainly concentrate on two respects. One is to determine the exact structure of $K(G)$ for some special kinds of graphs G [2,7,10,11,14–16]. The other is to study the relationship between the critical group of a graph and that of graphs obtained from it by various constructions. Maybe the most-known result in the latter respect is that the critical groups of a planar graph and its dual are isomorphic, which was first stated in [4], then rediscovered in [1,8], and generalized in [18]. Recently, the critical groups of line and directed line graphs are studied in [3,17] respectively. In particular, the authors of [3] showed that, for a regular non-bipartite graph, the critical group of the graph and that of its line graph determine each other uniquely in a simple fashion. Here, we shall use the same technique as in [3] to study the relations between the critical group of a graph and that of its clique-inserted graph. Following [29], we first give the formal definition of a clique-inserted graph.

Let v be a vertex of degree $d > 0$ in a graph G. If G' is the graph obtained from G by replacing the vertex v with a complete graph of order d (as illustrated in Fig. 1), then G' is said to be obtained by a d-clique insertion at v. Let G be a d-regular graph. We will call the graph operation of doing a d-clique insertion at every vertex of G clique inserting on G. The graph obtained

* Corresponding author.
E-mail addresses: chey5@jmu.edu.cn (H. Chen), fjzhang@xmu.edu.cn (F. Zhang).

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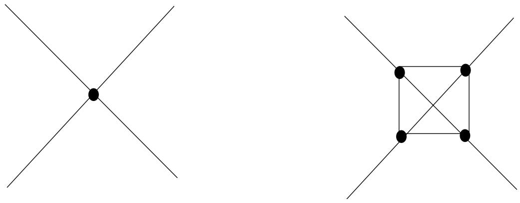

Fig. 1. The d-clique insertion at a vertex.

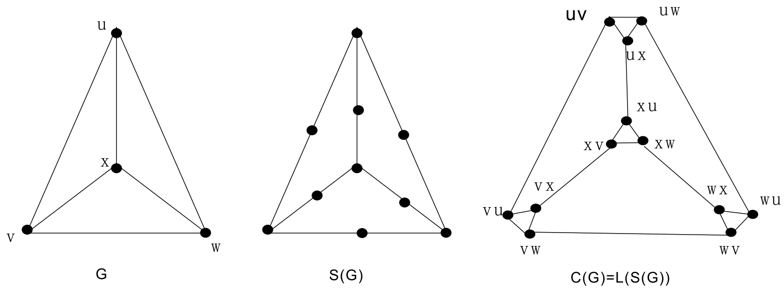

Fig. 2. $C(G)$ is the clique-inserted-graph of G.

from G by clique inserting will be called the *clique-inserted graph* of G [29] (or para-line graph [22]), denoted by $C(G)$. It is worth noting that $C(G)$ can also be viewed as the line graph of the subdivision of G (see Fig. 2).

Clearly, from any given d-regular graph, an infinite family of d-regular graphs is generated by applying clique-inserting iteratively. In particular, some interesting lattices in statistic physics, such as, the martini [21,24,26], the 3–12–12 [23–25], the 3–6–24 [13], the modified bathroom lattices [24], the Schreier graphs of the Hanoi Towers group [9], all can be constructed by clique-inserting. And Zhang [28] showed that new families of expander graphs can be constructed from the known ones in this way recently. So clique-inserting is an interesting transformation. Up to now, the relation between many parameters $C(G)$ and G had been considered, such as, the spectra, the tree-numbers, and the Kirchhoff indices. [22,27, 29,28]. Here we consider the relation between the critical group of $C(G)$ and that of G. First the tree-numbers of G and $C(G)$ have the following nice relation.

Theorem (*[29]*). *Let G be a connected simple d-regular graph with n vertices and m edges. Then*

$$\kappa(C(G)) = (d+2)^{m-n+1} d^{m-n-1} \kappa(G),$$

where $\kappa(G)$ is the number of spanning trees of G.

Since the order of the critical group $K(G)$ is just the tree-number of G. The above result suggests a close relationship between the critical group $K(G)$ and $K(C(G))$. The following theorem shows that in some sense it is indeed the case.

Theorem 1.1. *Let G be a connected simple d-regular ($d \geq 3$) graph with n vertices and m edges. Then there is a group homomorphism*

$$f : K(C(G)) \longrightarrow K(G)$$

whose kernel–cokernel exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow K(C(G)) \longrightarrow K(G) \longrightarrow \operatorname{coker}(f) \longrightarrow 0$$

has $\ker(f)$ *all $d(d+2)$-torsion, and*

$$\operatorname{coker}(f) = \begin{cases} 0, & \text{if } G \text{ is non-bipartite;} \\ \mathbb{Z}_d, & \text{if } G \text{ is bipartite.} \end{cases}$$

From the above theorem, it follows that:

Corollary 1.2. *Let G be a connected simple d-regular ($d \geq 3$) graph. If G is not bipartite, then $K(G)$ is isomorphic to a quotient group of $K(C(G))$.*