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On hyperbolic sets of maxes in dual polar spaces

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ABSTRACT

Suppose Δ is a fully embeddable thick dual polar space of rank $n \geq 3$. It is known that a hyperplane H of Δ is classical if all its nontrivial intersections with quads are classical. In order to conclude that a hyperplane H is classical, it is perhaps not necessary to require in advance that all these intersections are classical. In fact, in this paper we show that for dual polar spaces admitting hyperbolic sets of maxes, the existence of certain classical quad–hyperplane intersections implies that other quad–hyperplane intersections need to be classical as well. We will also derive necessary and sufficient conditions for two disjoint maxes to be contained in a (necessarily unique) hyperbolic set of maxes. Dual polar spaces admitting hyperbolic sets of maxes include all members of a class of embeddable dual polar spaces related to quadratic alternative division algebras.

1. Introduction

Let Π be a thick polar space of rank $n \geq 3$ (Tits [13, Chapter 7]). The maximal singular subspaces of Π then have (projective) dimension $n-1$. With Π, there is associated a dual polar space Δ of rank n. This is a point-line geometry whose points are the maximal singular subspaces of Π and whose lines are certain sets of maximal singular subspaces. Specifically, there exists a bijective correspondence between the singular subspaces of dimension $n-2$ of Π and the lines of Δ: if α is a singular subspace of dimension $n-2$, then the set L_α of all maximal singular subspaces containing α is a line of Δ. If F is a convex subspace of diameter $\delta \in \{2, 3, \dots, n\}$ of Δ, then we denote by $\widetilde{F}$ the point-line geometry induced on F by the lines of Δ that are contained in F. The geometry $\widetilde{F}$ is a dual polar space of rank δ. A convex subspace of diameter δ is called a *quad* if $\delta = 2$ and a *max* if $\delta = n-1$. The point-line geometry induced on a quad is a generalized quadrangle. There exists a bijective correspondence between the points of Π and the maxes of Δ. If x is a point of Π, then the set M_x of all maximal singular subspaces of Π containing x is a max of Δ. If M is a max of Δ, then we denote by x_M the unique point of Π corresponding to M.

A first class of objects under study in this paper are the hyperbolic sets of maxes. A set $\mathcal{H}$ of mutually disjoint maxes of Δ is called a *hyperbolic set of maxes* if the following two properties are satisfied:

(H1) every line of Δ meeting two distinct maxes of $\mathcal{H}$ meets all maxes of $\mathcal{H}$;
(H2) $L = \bigcup_{M \in \mathcal{H}} (M \cap L)$ for every line L of Δ meeting all maxes of $\mathcal{H}$.

In Section 2, we will determine necessary and sufficient conditions for two disjoint maxes of Δ to be contained in a (necessarily unique) hyperbolic set of maxes.

Hyperbolic sets of maxes have already been considered in the literature for symplectic dual polar spaces. In Section 3, we will indicate a larger class of dual polar spaces admitting hyperbolic sets of maxes. The dual polar spaces of this class are related to certain quadratic alternative division algebras.

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In Section 4, we will discuss an application of hyperbolic sets of maxes to hyperplanes of dual polar spaces. A *(full) projective embedding* of a point-line geometry $\mathcal{S}$ into a projective space Σ is an injective map e from the point set of $\mathcal{S}$ to the point set of Σ mapping lines of $\mathcal{S}$ to full lines of Σ such that the image of e generates the whole projective space Σ. A *hyperplane* of $\mathcal{S}$ is a set H of points, distinct from the whole point set, such that every line of $\mathcal{S}$ has either one or all its points in H. If $e : \mathcal{S} \to \Sigma$ is a full projective embedding of $\mathcal{S}$ and U is a hyperplane of the projective space Σ, then the set of all points of $\mathcal{S}$ that are mapped by e into U is a hyperplane of $\mathcal{S}$. Any hyperplane of $\mathcal{S}$ that can be obtained in this way is said to *arise from e*. A hyperplane of $\mathcal{S}$ is called *classical* if it arises from some full projective embedding. If H is a hyperplane of a dual polar space Δ and Q is a quad, then either $Q \subset H$ or $Q \cap H$ is a hyperplane of $\widetilde{Q}$. If $Q \subset H$, then the intersection $Q \cap H$ (which is equal to Q) is called *trivial*.

Combining results of Cardinali, De Bruyn & Pasini [2] and McInroy & Shpectorov [10] regarding simple connectedness of hyperplane complements in dual polar spaces and results of Ronan regarding hyperplanes and projective embeddings of point-line geometries (Corollaries 2 & 4 on page 180 and Corollary 4 on page 184 of [11]), we know that the following must hold (see [2] for more details):

Proposition 1.1 (*[2,10,11]*)**.** *Suppose Δ is a fully embeddable thick dual polar space. Then the following are equivalent for a hyperplane H of Δ:*

(1) *H is classical;*
(2) *for every quad Q of Δ not contained in H, the intersection $Q \cap H$ is a classical hyperplane of $\widetilde{Q}$.*

One can now wonder whether it is possible to prove a stronger version of Proposition 1.1 by relaxing condition (2). More precisely, one can wonder about the existence of a set $\mathcal{Q}$ of quads – not containing all quads and preferably as small as possible – such that Proposition 1.1 still remains valid if condition (2) is replaced by the following:

(2') for every quad $Q \in \mathcal{Q}$ not contained in H, the intersection $Q \cap H$ is a classical hyperplane of $\widetilde{Q}$.

In Section 4, we show that such sets $\mathcal{Q}$ exist if the embeddable dual polar space Δ admits hyperbolic sets of maxes. We show in this case that the existence of certain classical quad–hyperplane intersections implies that other quad–hyperplane intersections need to be classical as well. Among other things, we will prove the following in Section 4.

Proposition 1.2. *Let H be a hyperplane of a fully embeddable thick dual polar space Δ of rank at least 3, and let M_1 and M_2 be two disjoint maxes of Δ. Suppose M_1 and M_2 are contained in a (necessarily unique) hyperbolic set $\mathcal{H}$ of maxes. Suppose also that the following hold:*

(a) *For every $i \in \{1, 2\}$, $H_i := H \cap M_i$ is either M_i or a classical hyperplane of $\widetilde{M_i}$.*
(b) *For every quad Q meeting M_1 and M_2 (necessarily in lines), $H \cap Q$ is either Q or a classical hyperplane of $\widetilde{Q}$.*

Then for every $M \in \mathcal{H}$, we have that $H \cap M$ is either M or a classical hyperplane of $\widetilde{M}$.

2. Hyperbolic sets of maxes

Let Π denote a thick polar space of rank $n \geq 3$, and Δ its associated dual polar space. If A is a set of points of Π, then $A^\perp$ denotes the set of all points of Π collinear with all points of A. We also define $A^{\perp\perp} := (A^\perp)^\perp$. Two points of Δ are called *opposite* if they lie at maximal distance n from each other. Here, we follow the convention that distances $d(\cdot, \cdot)$ in Δ will always be measured in its collinearity graph. If x is a point and L a line of Δ, then L contains a unique point $\pi_L(x)$ nearest to x. Two lines of Δ are called *opposite* if they lie at maximal distance $n - 1$ from each other. If L_1 and L_2 are two opposite lines of Δ, then the maps $L_1 \to L_2; x \mapsto \pi_{L_2}(x)$ and $L_2 \to L_1; x \mapsto \pi_{L_1}(x)$ are bijections which are each other's inverses. If x_1 and x_2 are two points of Δ at distance δ from each other, then x_1 and x_2 are contained in a unique convex subspace $\langle x_1, x_2\rangle$ of diameter δ. If M is a max of Δ, then every point x not contained in M is collinear with a unique point $\pi_M(x)$ of M. If F is a convex subspace of diameter δ meeting a max M, then either $F \subseteq M$ or $F \cap M$ is a convex subspace of diameter $\delta - 1$.

Suppose M_1 and M_2 are two disjoint maxes of Δ. Then the map $M_1 \to M_2; x \mapsto \pi_{M_2}(x)$ is an isomorphism between $\widetilde{M_1}$ and $\widetilde{M_2}$. If x_1 and y_1 are two points of M_1 and if x_2 and y_2 denote the respective points of M_2 collinear with x_1 and y_1, then the distance between the lines $L_1 = x_1x_2$ and $L_2 = y_1y_2$ is equal to $d(x_1, y_1)$. Moreover, every point x of L_1 lies at distance $d(L_1, L_2)$ from a unique point of L_2, namely the point $\pi_{L_2}(x)$, and the maps $L_1 \to L_2; x \mapsto \pi_{L_2}(x)$ and $L_2 \to L_1; x \mapsto \pi_{L_1}(x)$ are bijections which are each other's inverses.

Lemma 2.1. *If L_1 and L_2 are two opposite lines of Δ, then $\{\langle u, \pi_{L_2}(u)\rangle \mid u \in L_1\}$ is a set of mutually disjoint maxes of Δ.*

Proof. Notice that if $u \in L_1$, then $d(u, \pi_{L_2}(u)) = n - 1$ and hence $\langle u, \pi_{L_2}(u)\rangle$ is a max.

Let u_1 and u_2 be two distinct points of L_1. Then $\pi_{L_2}(u_1) \neq \pi_{L_2}(u_2)$. Put $M_i := \langle u_i, \pi_{L_2}(u_i)\rangle$, $i \in \{1, 2\}$. Then $M_i \cap L_1 = \{u_i\}$, since every point of $L_1 \setminus \{u_i\}$ lies at distance n from $\pi_{L_2}(u_i)$. Suppose v is a point of $M_1 \cap M_2$. Since $\pi_{L_1}(v)$ is contained on a shortest path from $v \in M_1$ to $u_1 \in M_1$, we have $\pi_{L_1}(v) \in M_1$ and hence $\pi_{L_1}(v) = u_1$. A similar argument allows us to conclude that $\pi_{L_1}(v) \in M_2$ and $\pi_{L_1}(v) = u_2$. Since $u_1 \neq u_2$, this is not possible. So, the maxes M_1 and M_2 should be disjoint. ■