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Solving multivariate functional equations

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ABSTRACT

This paper presents a new method to solve functional equations of multivariate generating functions, such as

$$F(r,s) = e(r,s) + xf(r,s)F(1,1) + xg(r,s)F(qr,1) + xh(r,s)F(qr,qs),$$

giving a formula for $F(r,s)$ in terms of a sum over finite sequences. We use this method to show how one would calculate the coefficients of the generating function for parallelogram polyominoes, which is impractical using other methods. We also apply this method to answer a question from fully commutative affine permutations.

1. Introduction

Some generating functions are most naturally defined by a functional equation; in this article, we discuss a new method for solving certain functional equations involving multiple variables.

We set the stage with an elementary example: there are many families of combinatorial objects (e.g., binary trees, Dyck paths, triangulations of convex polygons; see [12]) whose generating function satisfies the functional equation

$$C(x) = 1 + xC(x)^2.$$

The generating function $C(x) = \sum_{n\geq 0} C_n x^n$ is a formal power series in one variable x that marks some statistic (often size) on the family of combinatorial objects. We can interpret the above equation to mean that every object in the family can either be represented as an object of size zero (an empty object) or as being composed of two smaller objects from the family. The solution to the above equation is

$$C(x) = \frac{1-\sqrt{1-4x}}{2x},$$

and the coefficients of the power series expansion of this generating function are the Catalan numbers. This gives a conceptual reason for the prevalence of Catalan numbers in combinatorics.

When we investigate multiple statistics at the same time, the functional equations can become more complicated. Suppose we are trying to solve for the generating function $F(s,t,x,y,q)$ that satisfies a functional equation of the type

$$F(s) = xe(s) + xf(s)F(1) + xg(s)F(qs). \tag{1.1}$$

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Here we have suppressed the variables t, x, y, and q to simplify the notation; the reader should interpret $F(sq)$ as $F(sq, t, x, y, q)$. In addition, the functions e, f, and g may be formal power series in all the variables. The combinatorial significance of a term like $F(sq)$ is that the statistic marked by q increases at the same time as the statistic marked by s as the objects are being built.

Functional equations of this type arise naturally when enumerating with statistics combinatorial objects such as polyominoes [6], plane trees [2], lattice paths [3], and pattern-avoiding permutations [4]. Bousquet-Mélou proved in [5, Lemma 2.3] that the solution to Eq. (1.1) is

$$F(s) = \frac{E(s) + E(1)G(s) - E(s)G(1)}{1 - G(1)},$$

where

$$E(s) = \sum_{n\geq 0} x^{n+1} g(s)g(sq)\cdots g(sq^{n-1})e(sq^n)$$

and

$$G(s) = \sum_{n\geq 0} x^{n+1} g(s)g(sq)\cdots g(sq^{n-1})f(sq^n).$$

Bousquet-Mélou used this lemma, as well as a generalization involving derivatives of F with respect to s, to find the generating function for various classes of column-convex polyominoes. Bousquet-Mélou and Brak's survey on enumerating polyominoes and polygons [6] is especially recommended reading.

Theorem 2.1 gives the solution to generating functions defined by a functional equation that simultaneously replaces multiple variables, such as in

$$F(r, s) = e(r, s) + xf(r, s)F(1, 1) + xg(r, s)F(qr, 1) + xh(r, s)F(qr, qs).$$

Our method applies to functional equations in an arbitrarily large number of formal variables r_1 through r_m and with arbitrarily many terms in the functional equation where each r_i can be replaced by 1 or $q^j r_i$ for $j \geq 0$.

Our proof is rather elementary—we repeatedly apply the functional equation and take the formal power series limit. Our main result (Theorem 2.1) is stated in terms of a sum over finite sequences. More importantly, in principle our method allows for the calculation of the coefficients of the generating function. This is in contrast to Bousquet-Mélou's result, which gives a quotient of q-Bessel functions, both of which are complicated infinite sums. Our method does not replace Bousquet-Mélou's method nor the powerful kernel method [1], which applies to many additional functional equations. We refer the reader interested in other solution methods to Bousquet-Mélou and Jehanne's [7].

In Section 2, we prove Theorem 2.1 and demonstrate how it applies to some simple functional equations including Eq. (1.1). In Section 3, we apply our method to the functional equation of parallelogram polyominoes and manipulate the solution to show how one would find the coefficients of the corresponding generating function. In Section 4, we apply a trivariate version of Theorem 2.1 to the study of fully commutative affine permutations, which was the original motivation for this study. An original analysis of fully commutative affine permutations in [9, Lemma 3.12] involves an unwieldy *ad hoc* calculation that we are able to replace by working it into the larger framework of solving multivariate functional equations.

2. Main result

We develop some notation in order to state our main result. Let $\mathbf{r}$ denote the set of formal variables $\{r_1, \ldots, r_m\}$. Our focus will be on $F(\mathbf{r})$, a formal power series in formal variables x, q, and $\mathbf{r}$, which is most easily described using a functional equation, as follows.

Throughout this article, we let $\mathbf{j} = (j_1, \ldots, j_m)$ denote an m-tuple of integers greater than or equal to -1 and let $\mathcal{J}$ be a set of such m-tuples $\mathbf{j}$. For each $\mathbf{j} \in \mathcal{J}$, let $f_{\mathbf{j}}(\mathbf{r})$ be a formal power series in the formal variables x, q, and $\mathbf{r}$. We also let $e(\mathbf{r})$ be a formal power series in x, q, and $\mathbf{r}$. We consider the functional equation

$$F(\mathbf{r}) = e(\mathbf{r}) + \sum_{\mathbf{j}\in\mathcal{J}} xf_{\mathbf{j}}(\mathbf{r})F\big((q^{j_1}r_1)^{\varepsilon(j_1)}, \ldots, (q^{j_m}r_m)^{\varepsilon(j_m)}\big), \tag{2.1}$$

where

$$\varepsilon(j) = \begin{Bmatrix} 1 & \text{if } j \geq 0 \\ 0 & \text{if } j = -1 \end{Bmatrix}.$$

This setup allows for arbitrarily many terms in the functional equation, where in each term, each variable r_i can be replaced by 1 (when $j_i = -1$) or $q^{j_i} r_i$ for $j_i \geq 0$.

We always let $J = (\mathbf{j}_1, \ldots, \mathbf{j}_n)$ denote a sequence of m-tuples, where each $\mathbf{j}_k$ is an integer sequence $\mathbf{j}_k = (j_{1,k}, \ldots, j_{m,k})$. For such a sequence J, we define two families of integers $a_{i,k}$ and $b_{i,k}$ for integers i and k satisfying $1 \leq i \leq m$ and $1 \leq k \leq n + 1$. First, define $a_{i,1} = 0$ and $b_{i,1} = 1$ for all i satisfying $1 \leq i \leq m$.

Then, for all i and k satisfying $1 \leq i \leq m$ and $2 \leq k \leq n + 1$, determine $a_{i,k}$ and $b_{i,k}$ in relation to the sequence $(j_{i,1}, j_{i,2}, \ldots, j_{i,k-1})$ consisting of the ith entry of the first $k - 1$ entries of J. If -1 occurs in this sequence, define $b_{i,k} = 0$ and $a_{i,k}$ to be the sum of the entries of the sequence after the last occurrence of -1; otherwise define $b_{i,k} = 1$ and set $a_{i,k}$ to be the sum of all entries in this sequence.