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Describing faces in plane triangulations

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Dedicated to Douglas R. Woodall on the occasion of his 70th birthday

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ABSTRACT

Lebesgue (1940) proved that every plane triangulation contains a face with the vertex-degrees majorized by one of the following triples:

$$(3, 6, \infty), (3, 7, 41), (3, 8, 23), (3, 9, 17), (3, 10, 14), (3, 11, 13),$$
$$(4, 4, \infty), (4, 5, 19), (4, 6, 11), (4, 7, 9), (5, 5, 9), (5, 6, 7).$$

Jendrol' (1999) improved this description, except for $(4, 4, \infty)$ and $(4, 6, 11)$, to

$$(3, 4, 35), (3, 5, 21), (3, 6, 20), (3, 7, 16), (3, 8, 14), (3, 9, 14), (3, 10, 13),$$
$$(4, 4, \infty), (4, 5, 13), (4, 6, 17), (4, 7, 8), (5, 5, 7), (5, 6, 6)$$

and conjectured that the tight description is

$$(3, 4, 30), (3, 5, 18), (3, 6, 20), (3, 7, 14), (3, 8, 14), (3, 9, 12), (3, 10, 12),$$
$$(4, 4, \infty), (4, 5, 10), (4, 6, 15), (4, 7, 7), (5, 5, 7), (5, 6, 6).$$

We prove that in fact every plane triangulation contains a face with the vertex-degrees majorized by one of the following triples, where every parameter is tight:

$$(3, 4, 31), (3, 5, 21), (3, 6, 20), (3, 7, 13), (3, 8, 14), (3, 9, 12), (3, 10, 12),$$
$$(4, 4, \infty), (4, 5, 11), (4, 6, 10), (4, 7, 7), (5, 5, 7), (5, 6, 6).$$

1. Introduction

The *degree* $d(v)$ of a vertex v ($r(f)$ of a face f) in a plane map M is the number of edges incident with it (loops are counted twice in $d(v)$, and cut-edges are counted twice in $r(f)$). By Δ and δ denote the maximum and minimum vertex degrees of M, respectively. A *k-vertex* (*k-face*) is a vertex (face) with degree k; a k^+*-vertex* has degree at least k, etc.

It is well known that each *normal* plane map, in which loops and multiple edges are allowed, but the degree of each vertex and face is at least three, has a 5^--vertex and a 5^--face. From now on, M denotes a normal plane map.

As proved by Steinitz [31], 3-polytopes are in 1–1 correspondence with 3-connected planar graphs. Plane triangulations are triangulated 3-polytopes; in particular, plane triangulations have neither loops nor multiple edges.

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The *weight* of a face in M is the degree-sum of its boundary vertices, and $w(M)$, or simply w, denotes the minimum weight of 5^--faces in M.

Let a face f be incident with vertices $x_1, \ldots, x_{r(f)}$, where $d(x_1) \le d(x_2) \le \cdots \le d(x_{r(x)})$. We say that f is a *face of type* $(k_1, \ldots, k_{r(f)})$, or simply a $(k_1, \ldots, k_{r(f)})$*-face*, where $k_1 \le \cdots \le k_{r(f)}$, if $d(x_1) = k_1$, $d(x_2) = k_2$, and $d(x_i) \le k_i$ whenever $3 \le i \le r(f)$. In other words, the boundary of a $(k_1, \ldots, k_{r(f)})$-face has a k_1-vertex, another vertex of degree k_2, yet another vertex of degree at most k_3, and so on. By a $(k_1, k_2^-, k_3, \ldots, k_{r(f)})$*-face* we mean a $(k_1, l_2, k_3, \ldots, k_{r(f)})$-face with $k_1 \le l_2 \le k_2$, etc.

Back in 1940, Lebesgue [23] gave an approximate description of 5^--faces in normal plane maps.

Theorem 1 (*Lebesgue [23]*)**.** *Every normal plane map has a* 5^-*-face of one of the following types:*

$(3, 6^-, \infty)$, $(3, 7, 41)$, $(3, 8, 23)$, $(3, 9, 17)$, $(3, 10, 14)$, $(3, 11, 13)$,

$(4, 4, \infty)$, $(4, 5, 19)$, $(4, 6, 11)$, $(4, 7, 9)$, $(5, 5, 9)$, $(5, 6, 7)$,

$(3, 3, 3, \infty)$, $(3, 3, 4, 11)$, $(3, 3, 5, 7)$, $(3, 4, 4, 5)$, $(3, 3, 3, 3, 5)$.

Theorem 1, along with other ideas in Lebesgue [23], has a lot of applications to plane graph coloring problems (first examples of such applications and a recent survey can be found in [8,28,30]).

Some parameters of Lebesgue's Theorem 1 were improved for certain subclasses of plane graphs. In 1963, Kotzig [21] proved that every plane triangulation with $\delta = 5$ satisfies $w \le 18$ and conjectured that $w \le 17$. In 1989, Kotzig's conjecture was confirmed by Borodin [2] in a more general form.

Theorem 2 (*Borodin [2]*)**.** *Every normal plane map with* $\delta = 5$ *has a* $(5, 5, 7)$*-face or a* $(5, 6, 6)$*-face, where all parameters are tight.*

Theorem 2 also confirmed a conjecture of Grünbaum [16] of 1975 that the cyclic connectivity (defined as the minimum number of edges to be deleted from a graph to obtain two components each containing a cycle) of every 5-connected planar graph is at most 11, which is tight (a bound of 13 was earlier obtained by Plummer [29]).

We note that a 3-polytope with $(4, 4, \infty)$-faces can have unbounded w, as follows from the n-pyramid. The same is true concerning $(3, 3, 3, \infty)$-faces: take the double $2n$-pyramid and delete all even upper spokes and all odd lower ones to obtain a quadrangulation having only $(3, 3, 3, 2n)$-faces.

For plane triangulations without 4-vertices, Kotzig [22] proved $w \le 39$, and Borodin [4], confirming Kotzig's conjecture in [22], proved $w \le 29$, which is best possible due to the dual of the twice-truncated dodecahedron. This was strengthened by Borodin [5] as follows: either there is a triangle of weight at most 17, or a triangle of weight at most 29 incident with a 3-vertex. Borodin [6] further shows that each triangulated 3-polytope without $(4^-, 4, \infty)$-faces satisfies $w \le 29$, and that for triangulations without $(4, 4, \infty)$-faces there is a sharp bound $w \le 37$.

Note that $29 = 3+5+21 = 3+6+20$, so already [4] implies that the terms $(3, 5, 21)$ and $(3, 6, 20)$ could be expected to appear in a tight description of faces in plane triangulations, where the sharpness of 20 in $(3, 6, 20)$ follows from the dual of the twice-truncated dodecahedron while the sharpness of 21 in $(3, 5, 21)$ is first established in the present paper (see Fig. 2). A similar remark concerns the tight term $(3, 4, 30)$ that comes from Borodin [6].

For arbitrary normal plane maps, Theorem 1 yields $w \le \max\{51, \Delta + 9\}$. Horňák and Jendrol' [17] strengthened this as follows: if there are neither $(4^-, 4, \infty)$-faces nor $(3, 3, 3, \infty)$-faces, then $w \le 47$. Borodin and Woodall [12] proved that forbidding $(3, 3, 3, \infty)$-faces implies $w \le \max\{29, \Delta + 8\}$.

Also, Horňák and Jendrol' [17] consider the minimum, w^*, of face weights over all faces instead of over only 5^--faces, as was being done before beginning with Lebesgue [23]. Clearly, $w^* \le w$. They proved [17] that any normal map avoiding $(4^-, 4, \infty)$-faces and $(3, 3, 3, \infty)$-faces satisfies $w^* \le 32$.

For quadrangulated 3-polytopes, Avgustinovich and Borodin [1] improved the description of 4-faces implied by Lebesgue's Theorem as follows: $(3, 3, 3, \infty)$, $(3, 3, 4, 10)$, $(3, 3, 5, 7)$, $(3, 4, 4, 5)$.

Some other results related to Lebesgue's Theorem can be found in the already mentioned papers, in a recent survey by Jendrol' and Voss [19], and also in [3,5,11–15,18,20,24–27,32].

In 2002, Borodin [7] strengthened nine parameters in Lebesgue's Theorem 1 without changing the others (the entries marked by an asterisk are best possible, see [7]).

Theorem 3 (*Borodin [7]*)**.** *Every normal plane map has a* 5^-*-face of one of the following types:*

$(3, 6^-, \infty^*)$, $(3, 7^*, 22)$, $(3, 8^*, 22)$, $(3, 9^*, 15)$, $(3, 10^*, 13)$, $(3, 11^*, 12)$,

$(4, 4, \infty^*)$, $(4, 5^*, 17)$, $(4, 6^*, 11)$, $(4, 7^*, 8)$, $(5, 5^*, 8)$, $(5, 6, 6^*)$,

$(3, 3, 3, \infty^*)$, $(3, 3, 4^*, 11)$, $(3, 3, 5^*, 7)$, $(3, 4, 4, 5^*)$, $(3, 3, 3, 3, 5^*)$.