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Note

Jeu-de-taquin promotion and a cyclic sieving phenomenon for semistandard hook tableaux

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ABSTRACT

Jeu-de-taquin promotion yields a bijection on the set of semistandard λ-tableaux with entries bounded by k. In this note, we determine the order of jeu-de-taquin promotion on the set of semistandard hook tableaux $CST((n-m, 1^m), k)$, with entries bounded by k, and on the set of semistandard hook tableaux with fixed content α, $CST((n-m, 1^m), k, \alpha)$. We give a bijection between $CST((n-m, 1^m), k, \alpha)$ and a suitable set of standard hook tableaux that behaves nicely with respect to jeu-de-taquin promotion and use the bijection to give a cyclic sieving phenomenon for $CST((n-m, 1^m), k, \alpha)$.

1. Introduction

Schützenberger's jeu-de-taquin promotion operator is a bijection on the set of standard λ-tableaux $SYT(\lambda)$ [11,12]. Haiman [5] showed that n iterations of jeu-de-taquin promotion fixes every standard rectangular tableau in $SYT(\lambda)$, where $\lambda = (c^r)$ is a partition of n. Rhoades [9] studied jeu-de-taquin promotion on column-strict, or semistandard, λ-tableaux $CST(\lambda, k)$, with entries bounded by k, and on semistandard λ-tableaux $CST(\lambda, k, \alpha)$ with a particular content α, and showed that the order of promotion on $CST((c^r), k)$ is equal to k unless the Young diagram of shape λ is a single row or column and $k = cr$. For general shapes, the order of promotion on $SYT(\lambda)$ and $CST(\lambda, k)$ is not known.

Rhoades also revealed a connection between jeu-de-taquin promotion and the cyclic sieving phenomenon (CSP) of Reiner, Stanton, and White [8]. Let X be a finite set, $C = \langle g \rangle$ a finite cyclic group of order N that acts on X, and $X(q)$ a polynomial with integer coefficients. The triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon (CSP) if, for any integer d,

$$X(\omega^d) = |\{x \in X \mid g^d \cdot x = x\}|,$$

where $\omega = e^{2\pi i/N}$ is a primitive Nth root of unity. Cyclic sieving phenomena have been exhibited in many different settings (see, for example, [1,4,6,7]).

Using Kazhdan–Lusztig theory, Rhoades [9] proved that $(SYT(c^r), C, X(q))$ exhibits the CSP, where $C = \langle j \rangle$ acts on $SYT(c^r)$ by jeu-de-taquin promotion and $X(q)$ is the q-analogue of the Frame–Robinson–Thrall hook-length formula [3]. Rhoades also gave a CSP for semistandard rectangular tableaux which involves the cyclic action of the jeu-de-taquin operator and the q-analogue of the hook-content formula. For a survey of the literature on the cyclic sieving phenomenon, see [10].

In Section 3, we define a bijection between the set of semistandard hook tableaux $CST((n-m, 1^m), k, \alpha)$, with entries bounded by k and fixed content α, and a set of standard λ'-tableaux, $SYT(\lambda')$, where the partition λ' is determined from

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$\lambda = (n-m, 1^m)$ and α. Our bijection behaves nicely with respect to jeu-de-taquin promotion, and we use it to prove that the order of promotion on $CST((n-m, 1^m), k, \alpha)$ is $p(\alpha) \cdot (n(\alpha) - 1)$, where $p(\alpha)$ is the cyclic symmetry of α and $n(\alpha)$ is the number of nonzero entries in α. We also prove that the order of promotion on $CST((n-m, 1^m), k)$ is equal to $k \cdot \mathrm{lcm}(m+1, m+2, \ldots, k-2, k-1)$ when $m+1 < k \leq n$ and $k \cdot \mathrm{lcm}(m+1, m+2, \ldots, n-2, n-1)$ when $k \geq n$.

In Section 4, we use our bijection combined with Reiner, Stanton, and White's CSP result for k-element subsets of a set $\{1, 2, \ldots, n\}$ to prove that $(X, C, X(q))$ exhibits the CSP, where C acts on $X = CST((n-m, 1^m), k, \alpha)$ by the $p(\alpha)$th power of promotion and $X(q)$ is the q-analogue of $\binom{n(\alpha)-1}{m}$.

2. Young tableaux and jeu-de-taquin promotion

An r-tuple of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is a *partition* of n if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. Throughout, λ shall be a fixed partition of a positive integer n.

The *Young diagram of shape* $\lambda = (\lambda_1, \ldots, \lambda_r)$ consists of r left-justified rows where the ith row contains λ_i boxes. A λ-*tableau* T is obtained by filling the boxes of the Young diagram of shape λ with positive integers. The entry in the ith row and jth column of T will be denoted by $T(i, j)$.

A λ-tableau is *semistandard*, or column strict, if the entries in its columns are strictly increasing from top to bottom and the entries in its rows are weakly increasing from left to right. We denote the set of semistandard λ-tableaux with entries less than or equal to a positive integer k by $CST(\lambda, k)$.

A k-tuple $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a *composition* of n of length k if $\alpha_i \geq 0$ for $1 \leq i \leq k$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$. A λ-tableau T has content $\alpha = (\alpha_1, \ldots, \alpha_k)$ if T has α_i entries equal to i for $1 \leq i \leq k$.

A λ-tableau is *standard* if its columns and its rows are strictly increasing and its content is equal to $\alpha = (1^n)$. We denote the set of standard λ-tableaux by $SYT(\lambda)$ and the set of semistandard λ-tableaux with content $\alpha = (\alpha_1, \ldots, \alpha_k)$ by $CST(\lambda, k, \alpha)$. Note that $SYT(\lambda) = CST(\lambda, n, (1^n))$. We now describe jeu-de-taquin promotion, which is a combinatorial algorithm that gives a function $\partial : CST(\lambda, k) \to CST(\lambda, k)$.

If $T \in CST(\lambda, k)$ does not contain entries equal to k, increase each entry in T by 1 to obtain $\partial(T)$. Otherwise, replace each k in T with a dot. If there is a dot in the figure that is not contained in a continuous strip of dots in the northwest corner, choose the westernmost dot and slide it north or west according to the following diagrams:

$$\begin{array}{|c|c|}\hline a & b \\ \hline \bullet \\ \cline{1-1}\end{array} \to \begin{array}{|c|c|}\hline \bullet & b \\ \hline a \\ \cline{1-1}\end{array}\,; \qquad \begin{array}{|c|c|}\hline a & \bullet \\ \hline b \\ \cline{1-1}\end{array} \to \begin{array}{|c|c|}\hline \bullet & a \\ \hline b \\ \cline{1-1}\end{array}\,; \qquad \begin{array}{|c|c|}\hline a & b \\ \hline c & \bullet \\ \hline\end{array} \to \begin{cases} \begin{array}{|c|c|}\hline a & \bullet \\ \hline c & b \\ \hline\end{array} & \text{if } c \leq b \\[2ex] \begin{array}{|c|c|}\hline a & b \\ \hline \bullet & c \\ \hline\end{array} & \text{if } b < c. \end{cases}$$

Continue sliding this dot north or west through the figure until it rests in a connected component of dots in the northwest corner. Repeat the procedure for the remaining dots in the figure until all dots belong to a connected component of dots in the northwest corner. Finally, replace all dots with 1s and increase all other entries in the figure by 1 to obtain $\partial(T)$. Jeu-de-taquin promotion preserves the semistandard property (see [13]), and the map $\partial : CST(\lambda, n) \to CST(\lambda, n)$ restricts to a function on $SYT(\lambda)$ which we will denote by $j : SYT(\lambda) \to SYT(\lambda)$.

The group generated by the long cycle $\sigma_k = (1, 2, \ldots, k) \in S_k$ acts on the set of compositions of n of length k by

$$\sigma_k(\alpha_1, \alpha_2, \ldots, \alpha_k) = (\alpha_{\sigma_k^{-1}(1)}, \alpha_{\sigma_k^{-1}(2)}, \ldots, \alpha_{\sigma_k^{-1}(k)}) = (\alpha_k, \alpha_1, \ldots, \alpha_{k-1}).$$

If T has content α, then $\partial(T)$ has content $\sigma_k \alpha$.

Example 2.1. Below is an illustration of jeu-de-taquin promotion on a tableau T in $CST((4, 4, 2), 7)$:

$$T = \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\ \hline 2&3&5&7\\ \hline 7&7\\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\ \hline 2&3&5&\bullet\\ \hline \bullet&\bullet\\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|c|}\hline \bullet&1&2&3\\ \hline 1&3&5&\bullet\\ \hline 2&\bullet\\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|c|}\hline \bullet&\bullet&2&3\\ \hline 1&1&5&\bullet\\ \hline 2&3\\ \cline{1-2}\end{array}$$

$$\to \begin{array}{|c|c|c|c|}\hline \bullet&\bullet&\bullet&3\\ \hline 1&1&2&5\\ \hline 2&3\\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|c|}\hline 1&1&1&4\\ \hline 2&2&3&6\\ \hline 3&4\\ \cline{1-2}\end{array} = \partial(T).$$

Given $T \in CST(\lambda, k)$, the *order of promotion* on T is the least positive integer r with $\partial^r(T) = T$. Given a subset X of $CST(\lambda, k)$ that is invariant under ∂, the *order of promotion* on X is the least positive integer r such that $\partial^r(T) = T$ for all T in X.