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Note

A note on distinct distances in rectangular lattices[☆]

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ABSTRACT

In his famous 1946 paper, Erdős (1946) proved that the points of a $\sqrt{n} \times \sqrt{n}$ portion of the integer lattice determine $\Theta(n/\sqrt{\log n})$ distinct distances, and a variant of his technique derives the same bound for $\sqrt{n} \times \sqrt{n}$ portions of several other types of lattices (e.g., see Sheffer (2014)). In this note we consider distinct distances in rectangular lattices of the form $\{(i,j) \in \mathbb{Z}^2 \mid 0 \le i \le n^{1-\alpha},\ 0 \le j \le n^{\alpha}\}$, for some $0 < \alpha < 1/2$, and show that the number of distinct distances in such a lattice is $\Theta(n)$. In a sense, our proof "bypasses" a deep conjecture in number theory, posed by Cilleruelo and Granville (2007). A positive resolution of this conjecture would also have implied our bound.

Given a set $\mathcal{P}$ of n points in $\mathbb{R}^2$, let $D(\mathcal{P})$denote the number of distinct distances that are determined by pairs of points from $\mathcal{P}$. Let $D(n) = \min_{|\mathcal{P}|=n} D(\mathcal{P})$; that is, $D(n)$ is the minimum number of distinct distances that any set of n points in $\mathbb{R}^2$ must always determine. In his celebrated 1946 paper [4], Erdős derived the bound $D(n) = O(n/\sqrt{\log n})$ by considering a $\sqrt{n} \times \sqrt{n}$ integer lattice (a variant of his technique derives the same bound for several other types of lattices; e.g., see Sheffer [11]). Recently, after 65 years and a series of progressively larger lower bounds,[1] Guth and Katz [8] provided an almost matching lower bound $D(n) = \Omega(n/\log n)$.

While the problem of finding the asymptotic value of $D(n)$is almost completely solved, hardly anything is known about which point sets determine a small number of distinct distances. Consider a set $\mathcal{P}$ of n points in the plane, such that $D(P) = O(n/\sqrt{\log n})$. Erdős conjectured [6] that any such set "has lattice structure". A variant of a proof of Szemerédi implies that there exists a line that contains $\Omega(\sqrt{\log n})$ points of $\mathcal{P}$ (Szemerédi's proof was communicated by Erdős in [5] and can be found in [9, Theorem 13.7]). A recent result of Pach and de Zeeuw [10] implies that any constant-degree curve that contains no lines and circles cannot be incident to more than $O(n^{3/4})$ points of $\mathcal{P}$. Another recent result by Sheffer, Zahl, and de Zeeuw [12] implies that no line can contain $\Omega(n^{7/8})$ points of $\mathcal{P}$, and no circle can contain $\Omega(n^{5/6})$ such points.

In this note we make some progress towards the understanding of the structure of such sets, by showing that rectangular lattices cannot have a sublinear number of distinct distances. Specifically, we consider the number of distinct distances that are determined by an $n^{1-\alpha} \times n^{\alpha}$ integer lattice, for some $0 < \alpha \le 1/2$. We denote this number by $D_\alpha(n)$.

The case $\alpha = 1/2$ is the case of the square $\sqrt{n} \times \sqrt{n}$ lattice, which determines $D_{1/2}(n) = \Theta(n/\sqrt{\log n})$ distinct distances, as already mentioned above. Surprisingly, we show here a different estimate for $\alpha < 1/2$.

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1 For a comprehensive list of the previous bounds, see [7] and http://www.cs.umd.edu/~gasarch/erdos_dist/erdos_dist.html (version of February 2014).

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Theorem 1. *For $\alpha < 1/2$, the number of distinct distances that are determined by an $n^{1-\alpha} \times n^{\alpha}$ integer lattice is*

$$D_\alpha(n) = n + o(n).$$

Proof. We consider the rectangular lattice

$$R_\alpha(n) = \{(i,j) \in \mathbb{Z}^2 \mid 0 \le i \le n^{1-\alpha},\ 0 \le j \le n^\alpha\}.$$

Notice that every distance between a pair of points of $R_\alpha(n)$ is also spanned by $(0, 0)$ and another point of $R_\alpha(n)$. This immediately implies $D_\alpha(n) \le n + O(n^{1-\alpha})$. In the rest of the proof we derive a lower bound for $D_\alpha(n)$. For this purpose, we consider the sublattice

$$R'_\alpha(n) = \{(i,j) \in \mathbb{Z}^2 \mid 2n^\alpha \le i \le n^{1-\alpha},\ 0 \le j \le n^\alpha\};$$

since $\alpha < 1/2$, $R'_\alpha(n) \ne \emptyset$ for $n \ge n_0(\alpha)$, a suitable constant depending on α. We also consider the functions

$$r(m) = \left|\{(i,j) \in R'_\alpha(n) \mid i^2 + j^2 = m\}\right|,$$
$$d(m) = \left|\{(i,j) \in R'_\alpha(n) \mid i^2 - j^2 = m\}\right|.$$

Observe that the smallest (resp., largest) value of m for which $d(m) \ne 0$ is $3n^{2\alpha}$ (resp., $n^{2-2\alpha}$).

We have the identities

$$\sum_m r(m) = \sum_m d(m), \tag{1}$$

$$\sum_m r^2(m) = \sum_m d^2(m). \tag{2}$$

The identity (1) is trivial. To see (2) we observe that the sum $\sum_m r^2(m)$ counts the number of ordered quadruples (i, j, i', j'), for $(i, j), (i', j') \in R'_\alpha(n)$, such that $i^2 + j^2 = i'^2 + j'^2$. But this quantity also counts the number of those ordered quadruples (i, j, i', j'), for $(i, j'), (i', j) \in R'_\alpha(n)$, such that $i^2 - j'^2 = i'^2 - j^2$, which is the value of the sum $\sum_m d^2(m)$. Putting (1) and (2) together we have

$$\sum_m \binom{r(m)}{2} = \sum_m \binom{d(m)}{2}. \tag{3}$$

Writing M_k for the set of those m with $r(m) = k$, we have $\sum_k k|M_k| = |R'_\alpha(n)|$. On the other hand,

$$\begin{aligned} D_\alpha(n) &\ge \sum_{k\ge1} |M_k| \\ &= \sum_{k\ge1} k|M_k| - \sum_{k\ge1}(k-1)|M_k| \\ &= |R'_\alpha(n)| - \sum_{k\ge2}(k-1)|M_k|. \end{aligned}$$

Thus $D_\alpha(n) \ge n - O(n^{2\alpha} + n^{1-\alpha}) - \sum_{k\ge2}(k-1)|M_k|$. Using the inequality $k - 1 \le \binom{k}{2}$ and (3), we have

$$\sum_{k\ge2}(k-1)|M_k| \le \sum_{k\ge2}\binom{k}{2}|M_k| = \sum_m \binom{r(m)}{2} = \sum_m \binom{d(m)}{2}.$$

Theorem 1 is therefore a trivial consequence of the following proposition.

Proposition 2.

$$\sum_m \binom{d(m)}{2} = O\left(n^{2\alpha}\log^2 n\right).$$

Proof. We need the following easy lemma.

Lemma 3. *If a positive integer m can be written as the product of two integers in two different ways, say $m = m_1m_2 = m_3m_4$, then there exists a quadruple of positive integers (s_1, s_2, s_3, s_4) satisfying*

$$m_1 = s_1s_2, \qquad m_2 = s_3s_4, \qquad m_3 = s_1s_3, \qquad m_4 = s_2s_4.$$

Proof. Since m_1 divides m_3m_4, we have $m_1 = s_1s_2$ for some $s_1 \mid m_3$ and some $s_2 \mid m_4$. Putting $s_3 = m_3/s_1$ and $s_4 = m_4/s_2$, we have $m_2 = s_3s_4$, $m_3 = s_1s_3$, and $m_4 = s_2s_4$. $\square$