Discrete Mathematics 336 (2014) 41–45

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

Decomposing the cube into paths

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ARTICLE INFO

Article history:
Received 28 October 2013
Received in revised form 25 July 2014
Accepted 26 July 2014
Available online 25 August 2014

Keywords:
Graph decomposition
Hypercube

ABSTRACT

We consider the question of when the n-dimensional hypercube can be decomposed into paths of length k. For odd n it is necessary that k divides $n2^{n-1}$ and that $k \leq n$. Anick and Ramras (2013) conjectured that these two conditions are also sufficient for all odd n and prove that this is true for odd $n \leq 2^{32}$. In this note we prove the conjecture.

1. Introduction

The *n-dimensional hypercube* $\mathcal{Q}_n$ is a graph with vertex set $V = \{0, 1\}^n$ and edge set $E = \{(x, y) : \|x-y\|_1 = 1\}$. Problems of decomposing the hypercube into edge-disjoint copies of smaller graphs have been considered by several authors, such as decomposing $\mathcal{Q}_n$ into trees [4,7], into Hamiltonian cycles and matchings [1] or into stars, $K_{1,r}$ for $r < n$ [3].

Mollard and Ramras [6], motivated by applications in parallel processing (see [5]), considered the problem of decomposing the hypercube into paths. A path of length k is a sequence of distinct vertices $x_1, x_2, \ldots, x_{k+1}$ such that for all $1 \leq i \leq k (x_i, x_{i+1}) \in E(\mathcal{Q}_n)$. Mollard and Ramras [6] noted that, if n is odd, and we wish to decompose $\mathcal{Q}_n$ into paths of length k, there are two simple necessary conditions that k must satisfy. Firstly, since $|E(\mathcal{Q}_n)| = n2^{n-1}$ we must have that k divides $n2^{n-1}$, which we write as $k \mid n2^{n-1}$. Secondly, since $\mathcal{Q}_n$ is n-regular, and n is odd, each vertex must be the endpoint of at least one of the paths, and so we must have at least 2^{n-1} paths (since each path has 2 endpoints). Therefore, we must also have that $k \leq n$. They were able to prove partial results towards a converse of this. For example they showed that if k and n are odd, and $k \mid n$, then $\mathcal{Q}_n$ can be decomposed into paths of length k, and if also $k < n$, then $\mathcal{Q}_n$ can be decomposed into paths of length $2k$. They also showed that $\mathcal{Q}_n$ can be decomposed into paths of length 4 for all $n \geq 4$. Anick and Ramras [2] conjectured:

Conjecture 1 ([2]). *Let n be odd and k such that $k \mid n2^{n-1}$ and $k \leq n$. Then $\mathcal{Q}_n$ can be decomposed into paths of length k.*

They showed that the conjecture holds for $n < 2^{32}$, a remarkably large bound. The main result of this note is to show that the conjecture holds for all n.

Theorem 2. *Let n be odd and k such that $k \mid n2^{n-1}$ and $k \leq n$. Then $\mathcal{Q}_n$ can be decomposed into paths of length k.*

In the next section we provide a proof of Theorem 2 and in the final section we briefly discuss what can be said about decomposing $\mathcal{Q}_n$ into paths of length k for even n.

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http://dx.doi.org/10.1016/j.disc.2014.07.019
0012-365X/

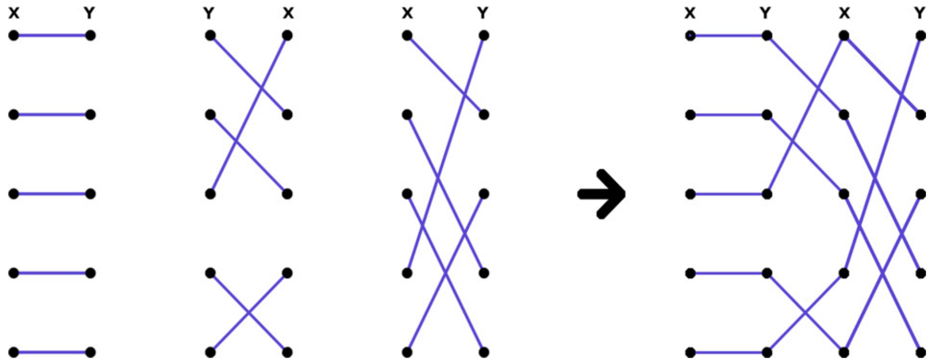

Fig. 1. Concatenation of 3 matchings, starting at X.

2. Proof of Theorem 2

A *walk of length k* is a sequence of vertices $x_1, x_2, \ldots, x_{k+1}$, not necessarily distinct, such that for all $1 \leq i \leq k$ $(x_i, x_{i+1}) \in E(\mathcal{Q}_n)$. We denote by *even vertices* the set of vertices $(q_1, q_2, \ldots, q_n) \in \mathcal{Q}_n$ such that $|\{i : q_i = 1\}|$ is even, and similarly *odd vertices*. It is apparent that $\mathcal{Q}_n$ is a bipartite graph, with the classes being the even and the odd vertices.

We will require a few lemmas for our proof. First, we will need the following lemma, which is a simple corollary of [2, Proposition 1].

Lemma 3. *For any n let t be such that t is odd and $t \mid n$. If $\mathcal{Q}_{\frac{n}{t}}$ can be decomposed into paths of length s then $\mathcal{Q}_n$ can be decomposed into paths of length ts.*

It follows from Lemma 3 that we only need to consider the case of decomposing $\mathcal{Q}_n$ into paths of length 2^r for $2^r \leq n$. Indeed, given some odd n, and $k \leq n$ such that $k \mid n$, we have that $k = t2^r$ for some odd $t \mid n$. So, by Lemma 3, if we can decompose $\mathcal{Q}_{\frac{n}{t}}$ into paths of length 2^r, then we can decompose $\mathcal{Q}_n$ into paths of length k. Note that $2^r \leq n/t$. We also note that for any graphs H, G_1 and G_2, it is a simple observation that, if G_1 and G_2 can be decomposed into copies of H, then so can the cartesian product $G_1 \times G_2$. Since we have that $\mathcal{Q}_{i+j} = \mathcal{Q}_i \times \mathcal{Q}_j$, the following lemma follows, which was also noted in [2, Lemma 4(b)]

Lemma 4. *If $\mathcal{Q}_i$ and $\mathcal{Q}_j$ can be decomposed into paths of length k then so can $\mathcal{Q}_{i+j}$.*

Finally, we will also need the following folklore result; for a proof see e.g. [1].

Lemma 5. *Let n be even. Then $\mathcal{Q}_n$ can be decomposed into edge-disjoint Hamiltonian cycles.*

Given any r, we want to show that $P_{2^r} \mid \mathcal{Q}_n$, for all odd $n \geq r$. The preceding two lemmas imply that, for each r, we only need to consider a finite number of n. The same result is shown in [2, Proposition 3].

Corollary 6. *If $P_{2^r} \mid \mathcal{Q}_{2^r+m}$ for all odd $1 \leq m \leq r + 1$, then $P_{2^r} \mid \mathcal{Q}_n$ for all odd $n \geq r$.*

Proof. Suppose first that r is odd. By Lemma 5 we have that $\mathcal{Q}_{r+1}$ can be decomposed into Hamiltonian cycles, of length 2^{r+1}, and so it can be decomposed into paths of length 2^r. We proceed by strong induction on n. The cases $2^r + 1 \leq n \leq 2^r + r + 1$ are assumed. Given $n \geq 2^r + r + 2$ we have that $n - (r + 1) \geq 2^r + 1$ and so by the induction hypothesis $P_{2^r} \mid \mathcal{Q}_{n-(r+1)}$. Hence, since $P_{2^r} \mid \mathcal{Q}_{r+1}$, by Lemma 4 $P_{2^r} \mid \mathcal{Q}_n$. The case where r is even is similar. □

A simple way to decompose $\mathcal{Q}_n$ into paths, which will inform our method, is as follows. Since $\mathcal{Q}_n$ is n-regular and bipartite, it is a simple application of Hall's theorem that we can decompose the edge set into n perfect matchings. Let X be the set of even vertices in $\mathcal{Q}_n$ and Y be the odd. If we take some perfect matchings $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k$, then we can cover the edges in these matchings by $|X|$ walks of length k, one starting at each vertex in X. For example, if the edge (x_1, y_{i_1}) is in $\mathcal{M}_1$ and the edge (y_{i_1}, x_{i_2}) is in $\mathcal{M}_2$ and so on, then we have that the walk starting at x_1 is $\{(x_1, y_{i_1}), (y_{i_1}, x_{i_2}), \ldots, (x_{i_{k-1}}, y_{i_k})\}$ if k is odd, and $\{(x_1, y_{i_1}), (y_{i_1}, x_{i_2}), \ldots, (y_{i_{k-1}}, x_{i_k})\}$ if k is even. We will use the notation $\mathcal{W}(\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k, X)$ to denote the set of walks formed by concatenating $\mathcal{M}_1$ to $\mathcal{M}_k$ in that order, starting at X, and similarly if we start at Y. A pictorial representation of this process is presented in Fig. 1.

Therefore, since as we noted before we can decompose $\mathcal{Q}_n$ into n perfect matchings, we can use this method to decompose $\mathcal{Q}_n$ into walks of length k, for any $k \mid n$, by splitting the matchings into sets of size k and concatenating them as above. If we are careful with the matchings we choose and the order we concatenate them in, we can ensure that these walks are paths. For example, if we take, for $1 \leq i \leq n$, the matchings

$$\mathcal{M}_i = \{((q_1, q_2, \ldots, q_i, \ldots, q_n), (q_1, q_2, \ldots, q_i + 1, \ldots, q_n)) : (q_1, q_2, \ldots, q_i, \ldots, q_n) \in X\}, \quad (2.1)$$