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Entire coloring of plane graph with maximum degree eleven☆

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ABSTRACT

A plane graph is called entirely k-colorable if for each $x \in V(G) \cup E(G) \cup F(G)$, we can use k colors to assign each element x a color such that any two elements that are adjacent or incident receive distinct colors. In this paper, we prove that if G is a plane graph with $\Delta = 11$, then G is entirely $(\Delta+2)$-colorable, which provides a positive answer to a problem posed by Borodin (Problem 5.2 in Borodin (2013)).

1. Introduction

Graphs considered in this note are finite, simple and undirected. Unless stated otherwise, we follow the notations and terminology in [1].

A plane graph is a particular drawing of a planar graph on the Euclidean plane. For a plane graph G, we use $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$ and $\delta(G)$ to denote, respectively, its *vertex set, edge set, face set, maximum degree* and *minimum degree*.

For $f \in F(G)$, we use $B(f)$ to denote the boundary walk of f and write $f = [u_1u_2 \cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the vertices of $B(f)$ in the clockwise order. A k (k^- or k^+)-*vertex* is a vertex of degree (at most or at least) k. A k (k^- or k^+)-*face* is defined similarly.

An *entire coloring* of a plane graph G is a coloring of the faces, vertices, and edges of G, which we call the elements of G, so that all incident or adjacent elements receive distinct colors. We use $\chi_{\text{vef}}(G)$ to denote the entire chromatic number of a plane graph G.

In 1972, Kronk and Mitchem [4] conjectured that any plane graph of maximum degree Δ is entirely $(\Delta + 4)$-colorable and proved this conjecture for $\Delta = 3$ [5]. In [6], it is proved that the conjecture is true if $\Delta \geq 6$. More recently, Wang and Zhu [9] completely settled the conjecture. In [9], the authors asked that whether every simple plane graph $G \neq K_4$ is entirely $(\Delta+3)$-colorable. Wang, Mao and Miao [8] proved that every plane graph with maximum degree $\Delta \geq 8$ is entirely $(\Delta + 3)$-colorable. It is obvious that $\chi_{\text{vef}}(G) \geq \Delta + 2$ for every plane graph. Borodin [2] proved that every plane graph G with $\Delta \geq 12$ satisfies $\chi_{\text{vef}}(G) = \Delta + 2$.

In this paper, we consider a problem posed by Borodin in [3], which states that: Is it true that $\chi_{\text{vef}}(G) \leq 13$ holds for every plane graph G with $\Delta = 11$?

Actually, we prove the following theorem and provide a positive answer to the above problem.

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Theorem 1.1. *If G is a plane graph with $\Delta = 11$, then $\chi_{\text{vef}}(G) \le \Delta + 2$.*

For convenience, we introduce the following terminology. A *partial (entire) coloring* is an entire coloring, except that some elements may not be colored. For any element x of G, we use $S(x)$ to denote the set of incident/adjacent elements of x.

Let v be a vertex of G with $N(v) = \{v_1, v_2, \dots, v_{d(v)}\}$ in clockwise order. The incident face bounded by $v_k v v_{k+1}$ is denoted by f_k for $0 \le k \le d(v) - 1$, where $k + 1$ is taken module $d(v)$. If there exist $v_i, v_j \in N(v)$ such that $\min\{d(v_i), d(v_j)\} \ge 3$ and $d(v_k) = 2$ for each $i + 1 \le k \le j - 1$, then we say that $\{v_i, \dots, v_j\}$ is a (v_i, v_j)-*segment* of v (or v contains a (v_i, v_j)-*segment*), denoted by $seg(v_i, v_j)$. It is needed to point out that $seg(v_i, v_j) \ne seg(v_j, v_i)$. The number of 2-vertices of $seg(v_i, v_j)$ is called the *span* of $seg(v_i, v_j)$, denoted by $|seg(v_i, v_j)|$. Two segments $seg(v_i, v_j)$ and $seg(v_k, v_l)$ are called adjacent if $v_j = v_k$ or $v_i = v_l$.

Let v be a vertex of G with $N(v) = \{v_1, v_2, \dots, v_{d(v)}\}$ in clockwise order and $f_k = v_k v v_{k+1}$ for some $1 \le k \le d(v)$, where k is taken module $d(v)$, be an incident face of v. If $d(v_k) \ge 3$ and $d(v_{k+1}) \ge 3$, then f_k is called a *non-segment face* of v.

Let v be a vertex of G, we use $n_i(v)$, $n_{i^+}(v)$ and $n_{i^-}(v)$ to denote the number of adjacent i, i^+ and i^--vertices of v. Analogously, the number of incident i, i^+ and i^--vertices of a face f is denoted by $n_i(f)$, $n_{i^+}(f)$ and $n_{i^-}(f)$.

2. Structures of counterexample

In this section, we always assume that G is a counterexample to Theorem 1.1 with $|V| + |E|$ as small as possible. In the following of this section, we present some structure properties of G.

Lemma 2.1. $\delta(G) \ge 2$.

Proof. To the contrary, suppose v is a 1-vertex of G and u is the neighbor of v. By the choice of G, $G - v$ has an entire coloring ϕ using 13 colors. Since $|S(v)| \le 12$, we can easily extend ϕ to the whole graph G. A contradiction.

Lemma 2.2. *If v is a 2-vertex of G and $N(v) = \{u, w\}$, then $d(u) \ge 10$ and $d(w) \ge 10$.*

Proof. Without loss of generality, we assume that $d(u) \le 9$. We contract uv to obtain a new graph G'. By the choice of G, G' admits an entire coloring ϕ. It is clear that ϕ is a partial coloring of G with v and uv uncolored. Since $d(v) = 2$ and $d(u) \le 9$, $S(uv) \le 13$ and $S(v) \le 6$. We can properly color uv and v in sequence to extend ϕ to the whole graph.

Lemma 2.3. *Suppose v is a 2-vertex of G and v is adjacent to a 10-vertex u. Let f_1 and f_2 be the two faces bounded by uv. Then $d(f_1) \ge 5$ and $d(f_2) \ge 5$. Moreover, if $d(f_i) = 5$, then $n_{4^+}(f_i) = 4$ for $i \in \{1, 2\}$.*

Proof. Let v be such a 2-vertex and let $N(v) = \{u, w\}$ and $d(u) = 10$. Without loss of generality, we assume that $d(f_1) \le 4$. We contract uv to obtain G'. By the minimality of G, G' admits an entire coloring ϕ using 13 colors. ϕ is a partial coloring of G with v and uv uncolored. First we erase the color of f_1. Notice that $|S(uv)| \le 14$, $|S(f_1)| \le 12$ and $|S(v)| \le 6$, we can properly color uv, f_1 and v in sequence to obtain an entire coloring of G.

Now we consider the latter part of the lemma. Assume that $d(f_1) = 5$ but $n_{4^+}(f_1) \le 3$. Let x be the other 3^--vertex incident with f_1.

The proof is quite similar to that of the previous. We contract uv to obtain G'. By the minimality of G, G' admits an entire coloring ϕ using 13 colors. ϕ is a partial coloring of G with v and uv uncolored. First we erase the color of f_1 and x. Notice that $|S(uv)| \le 14$, $|S(f_1)| \le 14$, $|S(x)| \le 9$ and $|S(v)| \le 6$, we can properly color uv, f_1, x and v in sequence to obtain an entire coloring of G.

Lemma 2.4. *Suppose v is a 2-vertex of G and v is adjacent to a 11-vertex u. Let f_1 and f_2 be the two faces bounded by uv with $d(f_1) \le d(f_2)$. If $3 \le d(f_1) \le 4$ or $d(f_1) = 5$ with $n_{4^+}(f_i) \le 3$, then $d(f_2) \ge 6$.*

Proof. Let v be a 2-vertex with $N(v) = \{u, w\}$ and $d(u) = 11$. To the contrary, we assume that $d(f_2) \le 5$. We contract uv to obtain G'. By the minimality of G, G' admits an entire coloring ϕ using 13 colors. ϕ is a partial coloring of G with v and uv uncolored. First we erase the color of f_1, f_2. Moreover, if $d(f_1) = 5$ with $n_{4^+}(f_i) \le 3$, we additionally erase the color of the other 3^--vertex of f_1, named x. Notice that $|S(uv)| \le 15$, $|S(f_1)| \le 14$, $|S(f_2)| \le 14$, $|S(x)| \le 9$ and $|S(v)| \le 6$, we can properly color $uv, f_2 f_1, x$ and v in sequence to obtain an entire coloring of G.

Lemma 2.5. *Let $f = [\cdots wuv \cdots]$ be a k-face of G with $3 \le k \le 4$. If $d(u) = 3$, then $d(v) \ge 10$ and $d(w) \ge 10$. If $d(u) \ge 4$, then $d(u) + d(v) \ge 12$.*

Proof. First we assume that $d(u) = 3$ and $d(v) \le 9$. By the choice of G, $G - uv$ admits an entire coloring ϕ. It is obvious that ϕ is a partial coloring of G with f and uv uncolored. We erase the color of v. Since $d(v) \le 9$, $|S(uv)| \le 14$. Notice that $|S(u)| \le 9$ and $|S(f)| \le 12$, we can properly color uv, f and u in sequence to obtain an entire coloring of G.

Now, we consider the latter part of the lemma. Assume that $d(u) \ge 4$ but $d(u) + d(v) \le 11$. By the choice of G, $G - uv$ admits an entire coloring ϕ with 13 colors. Notice ϕ is a partial coloring of G with f and uv uncolored. Since $d(u) + d(v) \le 11$, $|S(uv)| \le 13$. We can choose a proper color for uv, then properly color f to extend ϕ to the whole graph.

By Lemmas 2.2 and 2.5, we have Lemma 2.6.