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Nowhere-zero 3-flows of claw-free graphs

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ABSTRACT

Let A denote an abelian group and G be a graph. If a graph G^* is obtained by repeatedly contracting nontrivial A-connected subgraphs of G until no such a subgraph left, we say G can be A-reduced to G^*. A graph is claw-free if it has no induced subgraph $K_{1,3}$. Let $N_{1,1,0}$ denote the graph obtained from a triangle by adding two edges at two distinct vertices of the triangle, respectively. In this paper, we prove that if G is a simple 2-connected {claw, $N_{1,1,0}$}-free graph, then G does not admit nowhere-zero 3-flow if and only if G can be Z_3-reduced to two families of well characterized graphs or G is one of the five specified graphs.

1. Introduction

Graphs in this paper are finite, loopless, and may have multiple edges. We follow the notation and terminology in [1] except otherwise stated.

Let D be an orientation of a graph G. If an edge $e = uv \in E(G)$ is directed from a vertex u to a vertex v, then u is a *tail* of e and v is a *head* of e. For each vertex $v \in V(G)$, let $E^+(v)$ be the set of all directed edges with tails at v and $E^-(v)$ be the set of all directed edges with heads at v. Let A be an abelian group with identity 0, and let $A^* = A - \{0\}$. For every mapping $f : E(G) \to A$, the *boundary* of f is a function $\partial f : V(G) \to A$ defined by, for each $v \in V(G)$,

$$\partial f(v) = \sum_{e\in E^+(v)} f(e) - \sum_{e\in E^-(v)} f(e),$$

where "$\sum$" refers to the addition in A. A function f is an *A-flow* in G if $\partial f(v) = 0$ for each vertex $v \in V(G)$. A function f is a *nowhere-zero A-flow* if for each edge e, $f(e) \in A^*$. A graph G is *A-connected* if G has an orientation D such that for every function $b : V(G) \to A$ with $\sum_{v\in V(G)} b(v) = 0$, there exists a nowhere-zero A-flow f such that $\partial f = b$. For an integer $k \geq 2$, a *nowhere-zero k-flow* of G is an integer-valued function f on $E(G)$ such that $0 < |f(e)| < k$ for each $e \in E(G)$, and for each vertex $v \in V(G)$, $\partial f(v) = 0$. It is well known that G has a nowhere-zero Z_k-flow if and only if G has a nowhere-zero k-flow. As noted in [9], the existence of a nowhere-zero k-flow of a graph G is independent of the choice of the orientation D.

The concept of integer flow problems was introduced by Tutte in [20]. Group connectivity was introduced by Jaeger et al. in [9] as a generalization of nowhere-zero flows. The following conjectures are due to Tutte [20] and Jaeger et al. [9], respectively.

Conjecture 1.1. *Every 4-edge-connected graph admits a nowhere-zero 3-flow.*

Conjecture 1.2. *Every 5-edge-connected graph is Z_3-connected.*

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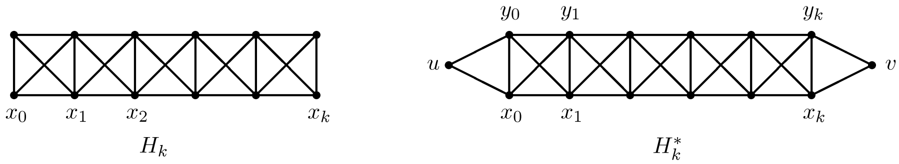

Fig. 1. Two families of graphs.

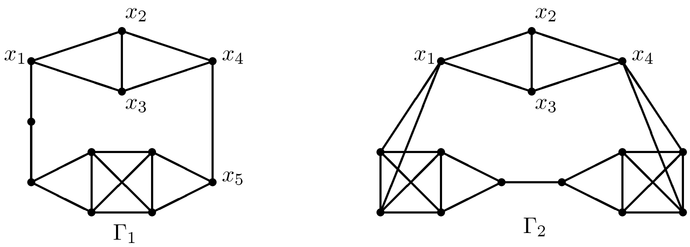

Fig. 2. Two graphs in $\mathscr{G}$.

Many authors have devoted themselves to the study of nowhere-zero flows. Some authors use degree condition to guarantee the existence of nowhere-zero 3-flows (see [5,6,24] and so on). Some authors investigated nowhere-zero flows on products of graphs (see [8,7,17] and so on). Some investigate nowhere-zero 3-flows in Cayley graphs (see [16,23] and others). Thomassen [19] confirmed the weak 3-flow conjecture by proving that every 8-edge-connected graph is Z_3-connected, which was recently improved to 6-edge-connected graph by Lovász et al. [14]. However, Conjectures 1.1 and 1.2 are still open. On the other hand, Xu [22] got the following observation.

Observation 1.3. Every 5-edge-connected graph is Z_3-connected if and only if every 5-edge-connected $K_{1,3}$-free graph is Z_3-connected.

Proof. It is obvious that if every 5-edge-connected graph is Z_3-connected, then every 5-edge-connected $K_{1,3}$-free graph is Z_3-connected. Conversely, assume that every 5-edge-connected $K_{1,3}$-free graph is Z_3-connected. Let G be a 5-edge-connected graph. Define G' be the graph obtained from G by replacing each vertex u of G with a complete graph $K_{\Delta(G)}$ such that no two neighbors of u are adjacent to the same vertex of the complete graph $K_{\Delta(G)}$. Since G is 5-edge-connected, $\Delta(G) \geq 5$. Thus, G' is 5-edge-connected $K_{1,3}$-free graph. By assumption, G' is Z_3-connected. By Proposition 3.2 and Corollary 3.5 of [11], G is Z_3-connected. ■

Conjecture 1.2 implies Conjecture 1.1 by a result of Kochol [10] that reduces Conjecture 1.1 to a consideration of 5-edge-connected graphs. In the view of Observation 1.3 and the result of Lovász et al. [14], both two conjectures reduce to investigate the Z_3-connectivity of 5-edge-connected $K_{1,3}$-free graphs. Thus, we investigate nowhere-zero 3-flows of $K_{1,3}$-free graphs in this paper. Before presenting our main result, we define some families of graphs.

For two disjoint graphs Γ_1 and Γ_2, a 2-*sum* of Γ_1 and Γ_2, denoted by $\Gamma_1 \oplus_2 \Gamma_2$, is the graph obtained from Γ_1 and Γ_2 by identifying one edge of Γ_1 with one edge of Γ_2. For a positive integer k, define H_k as follows. When $k = 1$, $H_1 = K_4$; for $k \geq 2$, denote by H_k the 2-sum of H_{k-1} and K_4 by identifying an edge $e = x_1x_2$ of H_{k-1} and $e' = y_1y_2$ of K_4, where $d_{H_{k-1}}(x_1) = d_{H_{k-1}}(x_2) = 3$. Define $\mathscr{H} = \{H_k | k = 2, 3, 4, \ldots\}$. Note that H_k contains exactly two pairs of adjacent vertices, each vertex of which is of degree 3. For $k \geq 1$, denote by H_k^* the graph obtained from H_k by adding two distinct vertices u and v, together with joining u to a pair of adjacent vertices of degree 3 in H_k and joining v to the other such pair of adjacent vertices in H_k. For $k \geq 1$, denote by H_k^{**} the graph obtained from H_k^* by deleting a vertex of degree 2 as depicted in Fig. 1.

Denote by K_n^- the graph obtained from a complete graph K_n by removing an edge. Define $\mathscr{A} = \{K_4^-, P_2\} \cup \{H_k^* : k = 1, 2, \ldots\}$. A vertex v of $H \in \mathscr{A}$ is a *distinguished vertex* if $d(v) = 2$ for $H \in \{K_4^-\} \cup \{H_k^* : k = 1, 2, \ldots\}$ or v is a vertex of P_2. Note that each graph in $\mathscr{A}$ has exactly two distinguished vertices. Let G be a 2-connected graph. An edge e of G is *essential* if $G - e$ is not 2-connected. Define $\mathscr{G}$ to be a family of graphs such that $G \in \mathscr{G}$ if and only if G is K_4 or G is a 2-connected graph such that

(1) G contains an essential edge e;
(2) each block of $G - e$ is a member of $\mathscr{A}$;
(3) the common vertex of two blocks B_1 and B_2 of $G - e$ is a distinguished vertex of B_1 and that of B_2 as well;
(4) $G - e$ contains at least one block isomorphic to K_4^- (see Fig. 2).

An edge is *contracted* if its two ends are identified into a single vertex and deleting the resulting loop. Let H be a connected subgraph of G. Denote by G/H the graph obtained from G by contracting all the edges of H and deleting all the resulting loops. A graph G is A-*reduced* if no nontrivial subgraph of G is A-connected. We say that a graph G_0 is an A-reduction of G if G_0 is A-reduced and if G_0 can be obtained from G by contracting all maximally A-connected subgraphs of G.