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Combinatorial proofs of five formulas of Liouville

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ABSTRACT

Liouville gave formulas for the number of representations of a positive integer by the quaternary quadratic forms $x^2+y^2+2z^2+2t^2$, $x^2+y^2+z^2+4t^2$, $x^2+y^2+4z^2+4t^2$, $x^2+4y^2+4z^2+4t^2$ and $x^2+2y^2+2z^2+4t^2$. These formulas have been proved by a number of authors by a variety of non-elementary methods. We give combinatorial proofs of these formulas by deducing them from Jacobi's four squares theorem and Legendre's four triangular numbers theorem. Since these latter two theorems have both been proved in an elementary arithmetic way, the five formulas of Liouville are therefore proved in a completely elementary way.

1. Introduction

Throughout this paper, let $\mathbb{N}_0$, $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of nonnegative integers, positive integers and integers, respectively. For positive integers a, b, c and d, we define

$$R(a,b,c,d;n)=\operatorname{card}\{(x,y,s,t)\mid ax^2+by^2+cs^2+dt^2=n,\ x,y,s,t\in\mathbb{Z}\}. \tag{1.1}$$

In 1829, Jacobi [6] discovered his famous formula for the number of representations of a positive integer as the sum of four squares, that is, for $n\in\mathbb{N}$,

$$R(1,1,1,1;n)=8\sigma(n)-32\sigma(n/4), \tag{1.2}$$

where $\sigma(n)$ is defined by

$$\sigma(n)=\sum_{\substack{d\mid n,\\ d\in\mathbb{N}}} d.$$

As usual, $\sigma(x)=0$ if x is a rational number but not a positive integer. An elementary arithmetic proof of (1.2) has been given by Spearman and Williams [14], see also [3].

In 1828, Legendre [7] established the formula for the number of representations of a nonnegative integer as the sum of four triangular numbers, namely, for $n\in\mathbb{N}_0$,

$$\operatorname{card}\left\{(x,y,s,t)\left|\frac{x(x+1)}{2}+\frac{y(y+1)}{2}+\frac{s(s+1)}{2}+\frac{t(t+1)}{2}=n,\ x,y,s,t\in\mathbb{N}_0\right.\right\}=\sigma(2n+1). \tag{1.3}$$

An elementary arithmetic proof of (1.3) has been given by Huard, Ou, Spearman and Williams [5], see also [3].

During the period 1859–1866, Liouville published more than 100 papers concerning representations of integers by more than 100 different quadratic forms. Five of them can be stated as follows.

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Theorem 1.1. *For $n \in \mathbb{N}$, we have*

$$R(1,1,2,2;n) = \begin{cases} 8\sigma(n/2) - 32\sigma(n/8), & n \equiv 0 \pmod 2, \\ 4\sigma(n), & n \equiv 1 \pmod 2. \end{cases} \tag{1.4}$$

This formula was stated by Liouville [8] and proved by Alaca, Alaca, Lemire and Williams [1], Bachmann [2], Deutsch [4] and Pepin [12,13].

Theorem 1.2. *For $n \in \mathbb{N}$, we have*

$$R(1,1,4,4;n) = \begin{cases} 8\sigma(n/4) - 32\sigma(n/16), & n \equiv 0 \pmod 4, \\ 4\sigma(n/2), & n \equiv 2 \pmod 4, \\ 0, & n \equiv 3 \pmod 4, \\ 4\sigma(n), & n \equiv 1 \pmod 4. \end{cases} \tag{1.5}$$

This formula was stated without proof by Liouville [9] and proved by Alaca, Alaca, Lemire and Williams [1], Bachmann [2] and Pepin [12,13].

Theorem 1.3. *For $n \in \mathbb{N}$, we have*

$$R(1,2,2,4;n) = \begin{cases} 8\sigma(n/4) - 32\sigma(n/16), & n \equiv 0 \pmod 4, \\ 4\sigma(n/2), & n \equiv 2 \pmod 4, \\ 2\sigma(n), & n \equiv 1 \pmod 2. \end{cases} \tag{1.6}$$

This form has been treated without proof by Liouville [11] and proved by Alaca, Alaca, Lemire and Williams [1], Bachmann [2] and Pepin [12,13].

Theorem 1.4. *For $n \in \mathbb{N}$, we have*

$$R(1,4,4,4;n) = \begin{cases} 8\sigma(n/4) - 32\sigma(n/16), & n \equiv 0 \pmod 4, \\ 0, & n \equiv 2, 3 \pmod 4, \\ 2\sigma(n), & n \equiv 1 \pmod 4. \end{cases} \tag{1.7}$$

This result was stated without proof by Liouville [10] and proved by Alaca, Alaca, Lemire and Williams [1] and Pepin [13].

Theorem 1.5. *For $n \in \mathbb{N}$, we have*

$$R(1,1,1,4;n) = \begin{cases} 12\sigma(n/2), & n \equiv 2 \pmod 4, \\ 8\sigma(n/4) - 32\sigma(n/16), & n \equiv 0 \pmod 4, \\ 6\sigma(n), & n \equiv 1 \pmod 4, \\ 2\sigma(n), & n \equiv 3 \pmod 4. \end{cases} \tag{1.8}$$

This formula was stated without proof by Liouville [10] and proved by Alaca, Alaca, Lemire and Williams [1] and Pepin [13].

Different methods were presented to prove Theorems 1.1–1.5, but to my knowledge, combinatorial proofs of these theorems have not been found. The goal of this paper is to provide combinatorial proofs of Theorems 1.1–1.5.

2. Combinatorial proof of Theorem 1.1

In this section, we give a combinatorial proof of Theorem 1.1. In order to prove Theorem 1.1, we first present the following three lemmas.

Lemma 2.1. *For $n \in \mathbb{N}_0$, we have*

$$\mathrm{card}\{(x,y) \mid x^2 + y(y+1) = n,\ x \in \mathbb{Z},\ y \in \mathbb{N}_0\} = \mathrm{card}\left\{(x,y) \mid \frac{x(x+1)}{2} + \frac{y(y+1)}{2} = n,\ x,y \in \mathbb{N}_0\right\}. \tag{2.1}$$

Proof. It is easy to see that for $n \in \mathbb{N}_0$,

$$\begin{aligned} \mathrm{card}\{(x,y) \mid x^2 + y(y+1) = n,\ x \in \mathbb{Z},\ y \in \mathbb{N}_0\} &= \mathrm{card}\{y \mid y(y+1) = n,\ y \in \mathbb{N}_0\} \\ &\quad + 2\,\mathrm{card}\{(x,y) \mid x^2 + y(y+1) = n,\ x \in \mathbb{N},\ y \in \mathbb{N}_0\} \end{aligned} \tag{2.2}$$