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Critical groups of covering, voltage and signed graphs

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ABSTRACT

Graph coverings are known to induce surjections of their critical groups. Here we describe the kernels of these morphisms in terms of data parametrizing the covering. Regular coverings are parametrized by voltage graphs, and the above kernel can be identified with a naturally defined voltage graph critical group. For double covers, the voltage graph is a signed graph, and the theory takes a particularly pleasant form, leading also to a theory of double covers of signed graphs.

1. Introduction

This paper studies graph coverings and critical groups for undirected *multigraphs* $G = (V, E)$; here E is a multiset of edges, with self-loops allowed. An example graph covering $\tilde{G} \to G$ is shown below, where the map sends an edge or vertex of $\tilde{G}$ to the corresponding edge or vertex of G by ignoring the $+/-$ subscript.

The *critical group* $K(G)$ is a subtle isomorphism invariant of G in the form of a finite abelian group, whose cardinality is the number of *maximal forests* in G. To present $K(G)$, one can introduce the *(signed) node–edge incidence* matrix $\partial := \partial_G$ for G having rows indexed by V, columns indexed by E, as we now explain. One defines ∂ by first fixing an arbitrary orientation of the edge set E. Then one lets the column of ∂ indexed by an edge e in E that has been oriented from vertex u to v be the difference vector $+u - v$, regarding each v in V as a standard basis vector for $\mathbb{R}^V$. One can regard ∂ as a map $\mathbb{Z}^E \to \mathbb{Z}^V$, and define $K(G)$ via either of these equivalent presentations (see Proposition 2.2)

$$K(G) := \operatorname{im}\partial / \operatorname{im}\partial\partial^t \quad (1.1)$$

$$\cong \mathbb{Z}^E / \left(\operatorname{im}\partial^t + \ker\partial\right) \quad (1.2)$$

where ∂^t is the map $\mathbb{Z}^V \to \mathbb{Z}^E$ corresponding to the transpose matrix of ∂. The presentation (1.1) allows one to compute the structure of $K(G)$ from the nonzero entries $d_1, d_2, \ldots, d_t$ in the Smith normal form of the *graph Laplacian matrix* $L(G) := \partial\partial^t$ appearing above:

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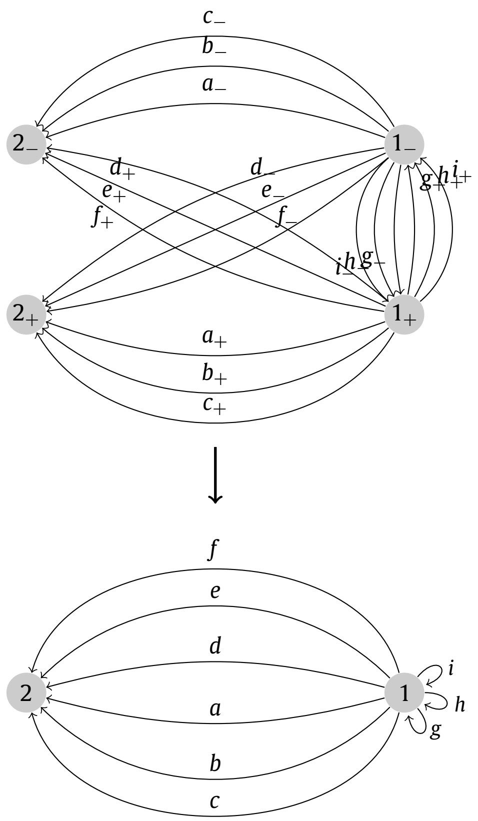

$$K(G) \cong \bigoplus_{i=1}^{t} \mathbb{Z}_{d_i}$$

where $\mathbb{Z}_d := \mathbb{Z}/d\mathbb{Z}$ denotes the cyclic group of order d.

Example. The graphs in the above covering $\tilde{G} \to G$ have node–edge incidence matrices

$$\partial_G = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{matrix} a & b & c & d & e & f & g & h & i \end{matrix} \\ \begin{pmatrix} +1 & +1 & +1 & +1 & +1 & +1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$\partial_{\tilde{G}} = \begin{array}{c} 1_+ \\ 1_- \\ 2_+ \\ 2_- \end{array} \begin{pmatrix} a_+ & b_+ & c_+ & a_- & b_- & c_- & d_+ & e_+ & f_+ & d_- & e_- & f_- & g_+ & h_+ & i_+ & g_- & h_- & i_- \\ +1 & +1 & +1 & 0 & 0 & 0 & 0 & 0 & 0 & +1 & +1 & +1 & +1 & +1 & +1 & -1 & -1 & -1 \\ 0 & 0 & 0 & +1 & +1 & +1 & +1 & +1 & +1 & 0 & 0 & 0 & -1 & -1 & -1 & +1 & +1 & +1 \\ -1 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

from which one obtains the Laplacian matrices

$$L(G) = \partial_G \partial_G^t = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{pmatrix} 6 & -6 \\ -6 & 6 \end{pmatrix} \end{array} \quad \text{and} \quad L(\tilde{G}) = \partial_{\tilde{G}} \partial_{\tilde{G}}^t = \begin{array}{c} \\ 1_+ \\ 1_- \\ 2_+ \\ 2_- \end{array} \begin{array}{c} \begin{matrix} 1_+ & 1_- & 2_+ & 2_- \end{matrix} \\ \begin{pmatrix} +12 & -6 & -3 & -3 \\ -6 & +12 & -3 & -3 \\ -3 & -3 & +6 & 0 \\ -3 & -3 & 0 & +6 \end{pmatrix} \end{array}$$

whose Smith normal forms

$$\begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 36 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

allow one to read off their critical groups:

$$K(G) \cong \mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3,$$