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Guides and shortcuts in graphs

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ABSTRACT

The geodesic structure of a graph appears to be a very rich structure. There are many ways to describe this structure, each of which captures only some aspects. Ternary algebras are for this purpose very useful and have a long tradition. We study two instances: signpost systems, and a special case of which, step systems. Signpost systems were already used to characterize graph classes. Here we use these for the study of the geodesic structure of a connected spanning subgraph F with respect to its host graph G. Such a signpost system is called a guide to (F, G). Our main results are: the characterization of the step system of a cycle, the characterization of guides for spanning trees and hamiltonian cycles.
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1. Introduction

The notion of distance in a graph G is a basic concept in graph theory. The distance $d(u, v)$ between two vertices u and v is the length of a shortest u, v-path or u, v-geodesic. Thus, V being the vertex set, (V, d) is an instance of a discrete metric space. As such there is a big difference with Euclidean space, that being a continuous metric space. But there is a more important difference: in a connected graph there may be many u, v-geodesics. The study of distance captures only one aspect of the set of geodesics. It turns out that the set of all geodesics represents a surprisingly rich structure with many subtleties. A classical tool to study this structure is the notion of interval: the interval $I(u, v)$ between u and v is the set of vertices on the u, v-geodesics. The first systematic study of the interval function I was [16]. An abundance of papers using the interval function has appeared since. Another way to study the geodesic structure is to use ternary algebras. Already in the early fifties of the last century Sholander [31–33] used ternary algebras to study betweenness, and using this Avann [2] studied graphs in 1961. By now there are many ways of using the algebraic approach to study the geodesic structure of a graph, not to mention other, non-algebraic approaches. We can only give a few examples here: [20,16,12,11,10,30,22,15,3,4]. None of these approaches by itself captures all aspects of the geodesic structure. So it appears that we need all these different approaches, and quite certainly even some more.

In this paper we single out two ternary algebras that capture different aspects of the geodesic structure: a *signpost system*, introduced in [18], and a special instance of signpost systems, the *step system of a graph*, introduced in [22]. See also [22, 26,28,29]. Loosely speaking a step in a graph is used to describe how to get one step closer from u to v. A signpost at u for getting to v directs us to a point x "closer to v", where we look for a new signpost that will lead us again closer to v. Note that a step system is defined on a graph, whereas a signpost system may be defined without reference to a graph. At first

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sight there are many similarities between the two systems. But a closer look reveals that the differences between the two systems allows us to capture different aspects of the geodesic structure.

A *ternary algebra* on a finite set V is a set $S \subseteq V \times V \times V$ of ordered triples from V that satisfies certain *axioms*. These axioms are such that they capture essential features of the structure to be studied. This approach is in the Sholander tradition of [31–33]. In various ways one can define the underlying graph of a ternary algebra, a tradition that probably originates with Avann [1,2]. Most of the papers on ternary algebras focus on graphs having an additional structure, such as median graphs and generalizations; see e.g. [2,21,16,13,5–7,18,14].

In Section 2 we study signpost systems and their underlying graphs. In this graph an edge ux is defined by the fact that (u, x, v) is a signpost, that is, to get to v from u our first step is to move to x. The underlying graph need not be connected. A point of focus is which axioms will guarantee that the underlying graph is in fact connected. We search for axioms that are in what we call standard form. The reason for this is the following. In [25] a striking and unexpected impossibility result was obtained: using first order logic the second author was able to prove that certain axiomatic characterizations of the induced path function do not exist. For this it was necessary to have the axioms in such standard form. See [17] for details of this problem.

In Section 4 we study a special type of signpost system that is not a step system. To explain the concepts of this section we use the following metaphor. Take a city, say Prague. The street plan of the old city with its overwhelming amount of beautiful historical buildings, sites, streets, bridges and squares may be represented by a connected graph G. A foreign tourist, say the first author, visits the city. Using the main touristic routes he will see many beautiful things, but he will miss many others and has to walk in crowded streets all day long. These main routes are represented by a spanning subgraph F of G. A local guide, say the second author, knows many shortcuts existing in G but not in F, to get from one place to another, and on the way many hidden treasures can be seen as well. Loosely speaking, a *guide* to the pair (F, G) is a signpost system that describes and studies these shortcuts. We believe that this concept of guide, that highlights the geodesic structure of a spanning subgraph with respect to its host graph, will shed new light on the study of the geodesic structure of graphs. In Sections 5 and 6 we study guides where the spanning subgraph is a tree or a hamiltonian cycle, respectively. As a preparation for the hamiltonian section we study the step system of a cycle in Section 3. This section can also be viewed independently as an instance of studying systematically the step systems of special classes of graphs; see [22,26] for other instances. In all the mentioned cases we obtain characterizations of the relevant signpost (step) systems involving various sets of axioms in standard form.

2. Signpost systems and their underlying graphs

Let V be a finite nonempty set. A *ternary system* $S = (V, R)$ on V consists of a set V and a ternary relation $R \subseteq V \times V \times V$ on V. We use the following convention: instead of $(v, w, x) \in R$ we write $vwSx$ and instead of $(v, w, x) \notin R$ we write $\neg vwSx$. Note that a similar convention was used in [28,29].

Let $G = (V, E)$ be a finite, simple, connected graph. For any u, v in V, we denote the geodesic *distance* between u and v by $d(u, v)$. It is the length of a shortest u, v-path, or u, v-*geodesic*. If w lies on a u, v-geodesic, then we say that w is *between* u and v. This way of viewing w as between u and v has been phrased in many different guises and languages. An important one is that of intervals: the *interval* between v and x in G is the set

$$I(v, x) = \{w \mid d(v, w) + d(w, x) = d(v, x)\},$$

in other words all vertices 'between' v and x; see [16,19]. Many other guises involve ternary systems, see e.g. [31–33,10,18,3,4]. We present two from the literature. The *geodesic betweenness* of G is the ternary system S defined as follows:

$$vwSx \quad \text{if and only if} \quad d(v, w) + d(w, x) = d(v, x).$$

Note that in this case v, w, x need not be distinct.

The *step system* of G is the ternary system S defined as follows:

$$vwSx \quad \text{if and only if} \quad d(v, w) = 1 \text{ and } d(w, x) = d(v, x) - 1.$$

This was introduced in [22]. Note that now v is necessarily distinct from w and x, but w and x may be the same vertex.

At first sight the two ternary systems seem to be quite similar. Of course this should be the case, because they both describe the same thing. But a closer look at the two systems reveals important differences. When one studies properties of both systems, especially, when one wants to characterize the betweenness relation and the step system using axioms on S, then axioms and proofs for both systems show essential differences. See [22] for step systems, and see e.g. [31,32,16,30,23,18,27,19] for various systems to describe the interval function or geodesic betweenness. We want to stress here the fact that the structure of the shortest paths in a graph is so rich that we need various ways to model this to be able to capture as many aspects of it as possible.

In [18] the notion of signpost system was introduced, which combines certain elements of the above two ternary systems, but is different from both. It captures yet some other aspects of the geodesic structure, see also [28,29]. Its name is derived from the fact that it reflects the structure of signposting in a road network. A *signpost system* is a ternary system $S = (V, R)$ satisfying the following three simple axioms:

(a1) if $vwSx$, then $wvSv$, for $v, w, x \in V$,

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