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Large vertex-transitive graphs of diameter 2 from incidence graphs of biaffine planes

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ABSTRACT

Under mild restrictions, we characterize all ways in which an incidence graph of a biaffine plane over a finite field can be extended to a vertex-transitive graph of diameter 2 and a given degree with a comparatively large number of vertices.

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1. Introduction

The best currently available construction of large vertex-transitive graphs of diameter 2 and a given degree [8] gives, for any integer $m \geq 1$, $\delta \in \{0, 1\}$, and any d of the form $d = 2^{2m+\delta} + (2+\delta)2^{m+1} - 6$, a Cayley graph of degree d, diameter 2, and order larger than $d^2 - 6\sqrt{2}d^{3/2}$. A slightly weaker result in terms of order but stronger in terms of symmetries was obtained in [10], where it is shown that for any odd prime power q and any ε with $0 < \varepsilon < 1$ there exists an infinite set of odd integers d of the form $d = q^3(q^{t-2}-1)/(q-1)$, where $t \geq 3/\varepsilon$ is an odd integer, for which there exists an arc-transitive graph of degree d, diameter 2, and order larger than $d^{2-\varepsilon}$. Both results can be seen as asymptotically achieving the Moore bound $d^2 + 1$ for degree d and diameter 2 by vertex-transitive graphs.

Since the degree sets in these two results are rather restricted, there still remains interest in producing families of large vertex-transitive graphs of diameter 2 for somewhat denser families of degrees. The most suitable starting graphs for this purpose still appear to be the McKay–Miller–Širáň graphs [4]. A geometric description of these graphs, given in [1,3], begins with a bipartite graph B_q of order $2q^2$, defined as follows. Let F be the Galois field of a prime power order q, with no further assumptions on q at this stage. The vertex set of B_q is $V_0 \cup V_1$, where $V_0 = \{(a,x)_0;\ a, x \in F\}$ and $V_1 = \{(b,y)_1;\ b, y \in F\}$, and the edge set $E(B_q)$ is given by $(a,x)_0 \sim (b, x+ab)_1$ for all $a, b, x \in F$. An alternative way to define B_q is to introduce it as the incidence graph of a biaffine plane of order q, which is an incidence structure obtained from a projective plane

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over F by removing a selected point and all lines through it and a selected line not incident with the selected point and all points on this line [2]. Letting $M_0(a) = \{(a, x)_0;\ x \in F\}$ and $M_1(b) = \{(b, y)_1;\ y \in F\}$, the McKay–Miller–Širáň graphs are then obtained by carefully adding edges to the sets $M_0(a)$ and $M_1(b)$. Without going into detail, it is proved in [4] that, if $q \equiv 1 \bmod 4$, edges in these sets can be added in a way to produce vertex-transitive graphs of degree $d = (3q-1)/2$ and diameter 2. In terms of d, the order of these graphs is $\frac{8}{9}(d+\frac{1}{2})^2$, which was until recently the closest value to the Moore bound achieved by vertex-transitive graphs. For alternative constructions of the McKay–Miller–Širáň graphs, we refer to [6].

This motivates the far more general problem of describing all 'reasonable' ways of extending the incidence graph B_q of a biaffine plane of order q to a vertex-transitive graph of diameter 2. To be more specific, observe that for every pair of distinct elements $a, a' \in F$ and for every $x, x' \in F$ the graph B_q contains a unique path of length 2 connecting the vertices $(a, x)_0$ and $(a', x')_0$. Similarly, for any two distinct $b, b' \in F$ and for any $y, y' \in F$ the graph B_q contains a unique path of length 2 with end vertices $(b, y)_1$ and $(b', y')_1$. It follows that in order to find a graph of diameter 2 whose spanning subgraph is B_q it would be superfluous to insert edges in V_0 joining pairs of different sets $M_0(a)$ and $M_0(a')$, as well as edges in V_1 between different sets $M_1(b)$ and $M_1(b')$.

In our contribution we will therefore be interested in describing *all* ways of adding edges within individual sets of the form $M_0(a)$ for $a \in F$ and $M_1(b)$ for $b \in F$ to produce a vertex-transitive graph of diameter 2. Formally, we say that a graph Γ is a *clustered extension* of B_q if Γ contains B_q as a spanning subgraph and the vertex set of each connected component of $\Gamma \setminus E(B_q)$ is a subset of $M_0(a)$ or $M_1(b)$ for $a, b \in F$. In these terms the task is as follows.

Problem. Characterize all vertex-transitive clustered extensions of B_q of diameter 2.

We will show that in most cases the solution to this problem can be pinned down to finding certain very specific elements, subsets, and automorphisms of the field F.

Theorem 1. *Let q be a prime power such that $q > 5$, and let δ be an integer such that $(q-1)/2 \le \delta \le q-1$. A vertex-transitive clustered extension of B_q with diameter 2 and degree $q+\delta$ exists if and only if the following condition $(*)$ is satisfied.*

$(*)$ *There exists a non-zero element $\tau \in F$, a subset $C \subset F \setminus \{0\}$, and an automorphism ϑ of F, such that $|C| = \delta$, $C = -C$, $C \cup \tau C^{\vartheta} = F \setminus \{0\}$, and $\tau\tau^{\vartheta} C^{\vartheta^2} = C$.*

This extends the findings of [3] regarding the McKay–Miller–Širáň graphs, generalizes some of the results of [9], and allows for new interesting constructions, but it also places severe restrictions on the ways in which clustered extensions can be constructed.

2. Proof of Theorem 1

We begin with a proof of necessity of $(*)$, which is the harder part. Let Γ be a clustered extension of B_q as in the statement of Theorem 1. Letting $\Gamma_0(a)$ and $\Gamma_1(b)$ be the subgraphs of Γ induced by the vertex sets $M_0(a)$ and $M_1(b)$ for $a, b \in F$, this means that the edge set of Γ is a disjoint union of the edge sets of the graphs $\Gamma_0(a)$ and $\Gamma_1(b)$, which are both regular of degree δ, together with the edge set of B_q. By the definition of the graph B_q there is a matching of size q between the q vertices of $M_0(a)$ and $M_1(b)$ formed by the edges $(a, x)_0 \sim (b, x+ab)_1$ for $x \in F$.

Suppose that the diameter of Γ is equal to 2, and take any $a, b \in F$. Our assumptions on Γ imply that the only way two vertices $(a, x)_0 \in M_0(a)$ and $(b, y)_1 \in M_1(b)$ with $y \ne x + ab$, can be connected by a path of length 2 is either by an edge $(a, x)_0 \sim (a, y-ab)_0$ followed by an edge $(a, y-ab)_0 \sim (b, y)_1$ of B_q, or by an edge $(a, x)_0 \sim (b, x+ab)_1$ of B_q followed by an edge of Γ of the form $(b, x+ab)_1 \sim (b, y)_1$. Since this is valid for any $x, y \in F$, letting $\phi:\ (a, x)_0 \mapsto (b, x+ab)_1$, we conclude that the union of the edge sets of the image $\phi(\Gamma_0(a))$ and of $\Gamma_1(b)$ must be the edge set of a complete graph on the vertex set $M_1(b)$. We will refer to this finding, valid for all $a, b \in F$, by loosely saying that "the union of $\Gamma_0(a)$ and $\Gamma_1(b)$ is a complete graph". A further obvious necessary condition, implied by the structure of Γ, is that all the subgraphs $\Gamma_0(a)$ and $\Gamma_1(b)$ for $a, b \in F$ have diameter 2. Conversely, these facts together with the existence of paths of length 2 in B_q discussed earlier imply the following.

Observation 1. *The graph Γ has diameter 2 if and only if, for any $a, b \in F$, the subgraphs $\Gamma_0(a)$ and $\Gamma_1(b)$ have diameter 2 and the union of $\Gamma_0(a)$ and $\Gamma_1(b)$ is a complete graph.*

Suppose now that the graph Γ of diameter 2 and degree $q+\delta$, where $\delta \ge (q-1)/2$, admits a vertex-transitive group of automorphisms G. By a simple counting, one sees that, if $\delta > q/2$, then for all $a, b \in F$ every edge in the subgraphs $\Gamma_0(a)$ and $\Gamma_1(b)$ lies in a triangle. But since no edge of Γ joining a vertex in V_0 with a vertex in V_1 lies in a triangle, we conclude that the group G acts on the vertex set of Γ with block system $\{V_0, V_1\}$ if $\delta > q/2$. We extend this observation also to the remaining two cases for δ if $q > 5$. For the rest of this paragraph, let $\delta = q/2$ for q a power of 2 and $q \ge 8$, or $\delta = (q-1)/2$ for odd $q > 5$. Suppose that $e = uv$ is an edge of $\Gamma' \in \{\Gamma_0(a), \Gamma_1(b)\}$ such that e is not contained in a triangle. This means that the vertex set of Γ' has the form $A \cup B \cup \{u, v\}$ if q is even, and $A \cup B \cup \{u, v\} \cup \{w\}$ if q is odd, where $|A| = |B| = \delta - 1$, the neighbourhood of u and v is $A \cup \{v\}$ and $B \cup \{u\}$, respectively, and w is joined neither to u nor to v if q is odd. Considering the fact that the degree of every vertex in A and in B is δ, we conclude the following. If q is even, then there are at least $(q-2)/2 > 2$ edges

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