Discrete Mathematics 314 (2014) 6–13

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Odd-cycle systems with prescribed automorphism groups

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ARTICLE INFO

Article history:
Received 3 December 2012
Received in revised form 24 August 2013
Accepted 9 September 2013
Available online 10 October 2013

Keywords:
Automorphism group
Cycle system

ABSTRACT

It is shown that for any finite group Γ, there exists a $(2k+1)$-cycle system whose full automorphism group is isomorphic to Γ. Furthermore the minimal order of such a system is at most $(4k+1)^{2\gamma \log_2 \gamma}$ if Γ is non-cyclic, and $(4k+1)^{3\gamma}$ otherwise, where $\gamma = |\Gamma|$.

1. Introduction

In 1975, E. Mendelsohn [9,10] proved that for any finite group Γ there exists a Steiner triple system with full automorphism group isomorphic to Γ. The order of the Steiner triple system in Mendelsohn's construction is less than $\gamma^{c\gamma}$, where $\gamma = |\Gamma|$ and c is a certain positive constant. In this paper, it is proved that for any finite group, Γ, there exists an odd-cycle system whose full automorphism group is isomorphic to Γ, and moreover our construction yields a $2k+1$-cycle system of order at most $(4k+1)^{2\gamma \log_2 \gamma}$ if Γ is non-cyclic, and $(4k+1)^{3\gamma}$ otherwise. This result was first presented in the author's doctoral thesis [8]. The thesis also contains the proof of the corresponding even cycles case, which has been accepted for publication [7]. More recently, a partial answer to the question of which abstract groups are the full automorphism groups of Hamiltonian cycle systems has been proved by Buratti, Rinaldi, and Traetta, [4].

An m-cycle system of order n, denoted by $m\mathrm{CS}(n)$, is a decomposition of the complete graph on n vertices, K_n, into cycles of length m. The vertices of K_n form the point set of the system. The order n is *m-admissible* if n is odd and $\frac{1}{2}n(n-1)$ is divisible by m. It has been shown [1,2,11] that an $m\mathrm{CS}(n)$ exists if and only if n is m-admissible and $n \geq m$. In this paper we shall be exclusively interested in cycles of odd length $2k+1$.

Following Mendelsohn, we shall use the result of Frucht [5] that for any given finite group Γ there exists a finite graph whose full automorphism group is isomorphic to Γ.

A *subsystem* of an $m\mathrm{CS}(n)$ is an $m\mathrm{CS}(n')$ with $n' \leq n$, all of whose m-cycles are also cycles of the $m\mathrm{CS}(n)$. A *proper* subsystem of an $m\mathrm{CS}(n)$ is a subsystem of order n' with $n' < n$.

To construct an $m\mathrm{CS}$ system with the same automorphism group as a given graph, we define a recursive construction analogous to a $|V|$-dimensional cube, where V is the vertex set of the graph. Subsystems on selected 2-faces of this cube are modified to give a new system with the required automorphism group.

Section 2 describes the recursive construction and derives some important properties. Section 3 details the modified construction, and proves that the modified system has the required automorphism group. Section 4 gives constructions which provide the basic building blocks required. Section 5 draws these elements together to prove the main result.

Before we proceed, we state several short results which will be needed later. The proofs of these results are given in [7].

Lemma 1.1. *If two subsystems of an $m\mathrm{CS}(n)$ have more than one point in common, then their common m-cycles also form a subsystem.*

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0012-365X/$ – see front matter
http://dx.doi.org/10.1016/j.disc.2013.09.006

Lemma 1.2. *If the set of fixed points of any automorphism of an* $m\mathrm{CS}(n)$ *has size greater than one, then it forms a subsystem.*

Lemma 1.3. *A cyclic* $m\mathrm{CS}(2m+1)$ *has no proper subsystems if* $m > 3$.

2. Basic construction

Given an $m\mathrm{CS}(n), S,\ (m, n \geq 3)$, and a non-zero positive integer v, we define an $m\mathrm{CS}(n^v)$ on the n^v points $(x_1, x_2, \ldots, x_v), x_i \in \{0, 1, \ldots, n-1\}$, which we denote by S^v. Before proceeding we define some notation. If $X = (x_1, x_2, \ldots, x_v)$ is any point of S^v we denote the value of the ith coordinate of X by $(X)_i$, i.e. $(X)_i = x_i$. We denote the set $\{0, 1, \ldots, n-1\}$ by the symbol $[n]$.

We now define the cycles of S^v. The sequence $(X_1, X_2, \ldots, X_m), X_j \in S^v, j = 1, 2, \ldots, m$ is a cycle of S^v if

(i) for at least one $i \in \{1, 2, \ldots, v\}$, the sequence

$$((X_1)_i, (X_2)_i, \ldots, (X_m)_i) \text{ is a cycle of } S \text{, and}$$

(ii) if $((X_1)_i, (X_2)_i, \ldots, (X_m)_i)$ is not a cycle of S, then

$$(X_1)_i = (X_2)_i = \cdots = (X_m)_i.$$

The cycles of S^v define an $m\mathrm{CS}(n^v)$, because if X_1, X_2 are distinct points of S^v, the edge (X_1, X_2) is in a unique cycle of S^v because for each $i \in \{1, 2, \ldots, v\}$ such that $(X_1)_i \neq (X_2)_i$, the edge $((X_1)_i, (X_2)_i)$ is in a unique cycle of S.

The construction just given is valid for all such S. However, from this point on we shall assume that S is has trivial automorphism group, and also that it has no proper subsystems.

The system S^v has subsystems of order n^k for all $1 \leq k < v$. We are particularly interested in the case $k = 1$, as we shall use these to derive the automorphism group for S^v. We characterise the subsystems of order n in the following lemma.

Lemma 2.1. *If* S *has a trivial automorphism group and has no proper subsystems, then each* $m\mathrm{CS}(n)$ *subsystem of* S^v *consists a set of points in which the values of* $k \geq 1$ *distinct coordinates* $i_1 < i_2 < \cdots < i_k$ *are equal, and take all the values in* $[n]$, *and the remainder of the coordinates are constant.*

Proof. Let T be any $m\mathrm{CS}(n)$ subsystem of S^v, and let $(X_1, X_2, \ldots, X_m)$ be any cycle in T. There is an i such that the set of points $\{(X_j)_i : j = 1, 2, \ldots, m\}$ has more than one element. The cycles obtained by taking the ith coordinate of the cycles of T forms a subsystem of S, since every pair of points is represented in some cycle. However this must be the whole of S, since S has no proper subsystems. Therefore, if the ith coordinate is not constant in T, it takes all values $0, 1, \ldots, n-1$.

If there are two distinct coordinates i, and j, whose values are both not constant for all points of T, then they both take all values $0, 1, \ldots, n-1$, and since T is of order n, each value of the ith coordinate occurs with only one value of the jth coordinate. These must be equal for all points of T, otherwise the correspondence would define a non-trivial automorphism of S. $\square$

Thus each $m\mathrm{CS}(n)$ subsystem of S^v is isomorphic to S. For each $A \subset \{1, 2, \ldots, v\}$, there are $n^{v-|A|}$ such $m\mathrm{CS}(n)$, one for each choice of the fixed values. We shall say that any two subsystems corresponding to the same set A are mutually *parallel*.

Lemma 2.2. *For each automorphism* ϕ *of* S^v *there is a permutation,* σ, *of* $1, \ldots, v$ *such that* $\phi(x_1, \ldots, x_v) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(v)})$, *for every* $x_1, \ldots, x_v \in [n]$.

Proof. We prove that for each $1 \leq i \leq v$, there is a $1 \leq j \leq v$ such that for any values $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_v \in [n]$, there are values $\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_v \in [n]$ with

$$\phi(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_v) = (\alpha_1, \ldots, \alpha_{j-1}, x, \alpha_{j+1}, \ldots, \alpha_v)$$

for each $x \in [n]$.

For any choice of i and of values $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_v$, the set of points $T = \{X(x) :\ (X(x))_r = a_r, r \neq i,\ (X(x))_i = x, x \in [n]\}$ forms an $m\mathrm{CS}(n)$ subsystem of S^v, and therefore so does the image of these points in ϕ. By Lemma 2.1 there is a non-empty set of values $A_i = \{j_1, j_2, \ldots, j_k\} \subset \{1, 2, \ldots, v\}$, and a set of values $\{\alpha_r :\ \alpha_r \in [n], r \in \{1, 2, \ldots, v\} \setminus A_i\}$ such that for each $X(x), x \in [n]$ there is a $y \in [n]$ with $(\phi(X(x)))_r = y$ for $r \in A_i$ and $(\phi(X(x)))_r = \alpha_r$ otherwise.

Observe that for any $r \in A_i$, the mapping $x \mapsto X(x) \mapsto (\phi(X(x)))_r = y$ provides an automorphism of S, from which we conclude that $y = x$, since S has a trivial automorphism group.

We now show that the images under ϕ of the $m\mathrm{CS}(n)$ subsystems obtained by varying the $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_v$ are mutually parallel.

In order to show this it is only necessary to show that A_i remains the same if we change any one $a_t, t \neq i$ at a time. So choose $t \neq i$, $1 \leq t \leq v$, and substitute any value $a'_t \neq a_t$ for a_t. Then there is a set $A'_i \subset \{1, 2, \ldots, v\}$, and values $\alpha'_r, r \in \{1, 2, \ldots, v\} \setminus A'_i$ such that ϕ maps each point $X'(x) = (a_1, \ldots, a_{t-1}, a'_t, a_{t+1}, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_v)$ to the point $\phi(X'(x))$, where $(\phi(X'(x)))_r = x$ if $r \in A'_i$, and $(\phi(X'(x)))_r = \alpha'_r$ otherwise.