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Spanning k-ended trees of bipartite graphs

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ABSTRACT

A tree is called a k-ended tree if it has at most k leaves, where a leaf is a vertex of degree one. We prove the following theorem. Let $k \geq 2$ be an integer, and let G be a connected bipartite graph with bipartition (A, B) such that $|A| \leq |B| \leq |A| + k - 1$. If $\sigma_2(G) \geq (|G| - k + 2)/2$, then G has a spanning k-ended tree, where $\sigma_2(G)$ denotes the minimum degree sum of two non-adjacent vertices of G. Moreover, the condition on $\sigma_2(G)$ is sharp. It was shown by Las Vergnas, and Broersma and Tuinstra, independently that if a graph H satisfies $\sigma_2(H) \geq |H| - k + 1$ then H has a spanning k-ended tree. Thus our theorem shows that the condition becomes much weaker if a graph is bipartite.

1. Introduction

In this paper, we consider finite simple graphs, which have neither loops nor multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We write $|G|$ for the order of G, that is, $|G| = |V(G)|$. For a vertex v of G, let $N_G(v)$ denote the neighborhood of v in G, and denote the degree of v in G by $\deg_G(v)$, in particular, $\deg_G(v) = |N_G(v)|$. For two vertices x and y of G, an edge joining them is denoted by xy or yx. A vertex of a tree is called a *leaf* if its degree is one. For an integer $k \geq 2$, a tree is called a *k-ended tree* if it has at most k leaves.

The invariant $\sigma_2(G)$ is defined to be the minimum degree sum of two non-adjacent vertices of G, i.e.,

$$\sigma_2(G) = \min_{xy \notin E(G)} \{\deg_G(x) + \deg_G(y)\}.$$

By using $\sigma_2(G)$, Ore obtained the following famous theorem on Hamilton path. Notice that a Hamilton path is a spanning 2-ended tree.

Theorem 1 (*Ore [7]*). *Let G be a connected graph. If $\sigma_2(G) \geq |G| - 1$, then G has a Hamilton path.*

The following theorem gives a similar sufficient condition for a graph to have a spanning k-ended tree.

Theorem 2 (*Las Vergnas [4], Broersma and Tuinstra [2]*). *Let $k \geq 2$ be an integer, and let G be a connected graph. If $\sigma_2(G) \geq |G| - k + 1$, then G has a spanning k-ended tree.*

Our main result of this paper is the following theorem, which shows that the lower bound on $\sigma_2(G)$ in Theorem 2 can be much weakened for bipartite graphs.

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Theorem 3. *Let $k \geq 2$ be an integer, and let G be a connected bipartite graph with bipartition (A, B) such that $|A| \leq |B| \leq |A| + k - 1$. If*

$$\sigma_2(G) \geq \frac{|G| - k}{2} + 1, \tag{1}$$

then G has a spanning k-ended tree.

Note that the condition $|B| \leq |A| + k - 1$ is necessary for the bipartite graph G to have a spanning k-ended tree. Moreover, the degree sum condition is sharp in the sense that we cannot replace the lower bound on $\sigma_2(G)$ by $(|G| - k + 1)/2$. We show this sharpness in the last section.

On the other hand, one might conjecture that $\sigma_2(G)$ can be replaced by

$$\sigma_{1,1}(G) = \min_{xy \notin E(G)} \{\deg_G(x) + \deg_G(y) \mid x \in A,\ y \in B\}.$$

In fact, we obtain the following theorem.

Theorem 4. *Let $k \geq 2$ be an integer, and let G be a connected bipartite graph with bipartition (A, B) such that $|A| \leq |B| \leq |A| + k - 1$. If*

$$\sigma_{1,1}(G) \geq |B|, \tag{2}$$

then G has a spanning k-ended tree.

The above Theorem 4 is a generalization of the following Theorem 5 on Hamilton path.

Theorem 5 (*Moon and Moser [5]*)**.** *Let G be a connected bipartite graph with bipartition (A, B) such that $|A| \leq |B| \leq |A| + 1$. If $\sigma_{1,1}(G) \geq |B|$, then G has a Hamilton path.*

Many results on spanning k-ended trees related to our theorems can be found in the book [1] and papers [3,6] and so on. In particular, a survey article [8] contains many current results on spanning trees including spanning k-ended trees.

2. Proof of Theorem 3

We begin with some notation. Let G be a graph. A set X of vertices of G is called an independent set if no two vertices of X are adjacent in G. Let T be a tree. We denote the set of leaves of T by $Leaf(T)$. For two vertices u and v of T, there exists a unique path connecting u and v in T, and it is denoted by $P_T(u, v)$. We need the next lemma.

Lemma 2.1. *Let T be a tree whose vertices are colored with red and blue so that no two adjacent vertices have the same color. If all the leaves of T are red, then the number of red vertices of T is greater than or equal to the number of blue vertices of T.*

Proof. We prove the lemma by induction on $|T|$. It is easy to see that the lemma holds for small trees. Let T be a tree. We remove all the leaves from T, and denote the resulting tree by T_1. Again remove all the leaves of T_1 from T_1, and denote the resulting tree by T_2. Since every leaf of T_1 is adjacent to at least one leaf of T, the number of leaves of T is greater than or equal to the number of leaves of T_1. It is easy to see that all the leaves of T_2 are red. Hence by the induction hypothesis, the number of red vertices of T_2 is at least the number of blue vertices of T_2. Therefore the lemma holds. □

We now prove Theorem 3.

Proof of Theorem 3. Let G be a connected bipartite graph with bipartition (A, B) that satisfies all the conditions in Theorem 3. Suppose that G has no spanning k-ended tree. We choose a spanning tree T of G so that

(T1) the number of leaves of T is as small as possible;
(T2) the length of a longest path in T is as large as possible subject to (T1).

Then the number of leaves of T is $|Leaf(T)| = \ell \geq k + 1 \geq 3$ since G has no spanning k-ended tree. In particular, T is not a Hamilton path and has at least one vertex of degree at least three.

For convenience, we call a vertex of B a *red vertex* and a vertex of A a *blue vertex*. We shall consider two cases.

Case 1. *T contains both a red leaf and a blue leaf.*

Let v be a red leaf and w a blue leaf of T. It is easy to see that no two leaves of T are adjacent in G since otherwise we can get a spanning tree having fewer leaves than T. For a vertex $x \in V(T) - \{v, w\}$, x_v denotes the vertex which is adjacent to x and lies on the path $P_T(x, v)$, and x_w is defined analogously. In other words, x_v (x_w) is the parent of x in a rooted tree T with root v (w), respectively. So if x does not lie on $P_T(v, w)$, then $x_v = x_w$, and if x lies on $P_T(v, w)$, then $x_v \neq x_w$.

Suppose that v is adjacent to a vertex x in G but not in T, and that w is adjacent to x_v in G. Then x is not a leaf of T, and $T_1 = T - xx_v + vx + wx_v$ contains a unique cycle, which has an edge e_1 incident with a vertex of degree at least three in T_1. Then $T_1 - e_1$ is a spanning tree of G with $\ell - 1$ leaves. This contradicts the choice (T1) of T. Hence w is not adjacent to x_v in G. If v is adjacent to a vertex x in T, then $x_v = v$ and so w is not adjacent to x_v in G.