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Note

A game generalizing Hall's Theorem

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ABSTRACT

We characterize the initial positions from which the first player has a winning strategy in a certain two-player game. This provides a generalization of Hall's Theorem. Vizing's Theorem on edge-coloring follows from a special case.

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1. Introduction

A *set system* is a finite family of finite sets. A *transversal* of a set system $\mathcal{S}$ is an injection $f : \mathcal{S} \hookrightarrow \bigcup \mathcal{S}$ such that $f(S) \in S$ for each $S \in \mathcal{S}$. Hall's Theorem [4] gives the precise conditions under which a set system has a transversal.

Theorem 1.1 (*Hall [4]*). *A set system $\mathcal{S}$ has a transversal if and only if $\left|\bigcup \mathcal{W}\right| \geq |\mathcal{W}|$ for each $\mathcal{W} \subseteq \mathcal{S}$.*

We generalize this by analyzing winning strategies in a two-player game played on a set system by *Fixer* (henceforth dubbed **F**) and *Breaker*. Fixer wins the game by eventually modifying the set system so that it has a transversal; if Breaker has a strategy to prevent this forever, then we say that Breaker wins. Additionally, when playing on the set system $\mathcal{S}$, we provide a *pot* P with $\bigcup \mathcal{S} \subseteq P$. Fixer moves first and he can do the following.

Fixer's turn. Pick $x \in P$ and $S \in \mathcal{S}$ with $x \notin S$ and replace S with $S \cup \{x\} \smallsetminus \{y\}$ for some $y \in S$.

For $k \in \mathbb{N}$, let $[k] = \{1, \dots, k\}$. For each $t \in [|\mathcal{S}| - 1]$, we have a different rule for Breaker. We denote Breaker by $\mathbf{B}_t$ when he is playing with the following rule.

Breaker's turn. If **F** modified $S \in \mathcal{S}$ by inserting x and removing y, $\mathbf{B}_t$ can pick up to t sets in $\mathcal{S} \smallsetminus \{S\}$ and modify them by swapping x for y or y for x.

To state the main theorem, we need additional notation. For $\mathcal{W} \subseteq \mathcal{S}$ and $x \in P$ define the *degree* in $\mathcal{W}$ of x, written $d_{\mathcal{W}}(x)$, by

$$d_{\mathcal{W}}(x) = \left|\{S \in \mathcal{W} : x \in S\}\right|.$$

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Define the *t-value* of $\mathcal{W} \subseteq \mathcal{S}$, written $\nu_t(\mathcal{W})$, by

$$\nu_t(\mathcal{W}) = \sum_{x \in \bigcup \mathcal{W}} \left\lfloor \frac{d_{\mathcal{W}}(x) - 1}{t+1} \right\rfloor.$$

Intuitively, this measures how much **F** can increase $\left|\bigcup \mathcal{W}\right|$ without $\mathbf{B}_t$ undoing the progress. For instance, if $d_{\mathcal{W}}(y) \le t+1$ and **F** swaps x in for y at W, then B_t can change all instances of x to y, since x appears in at most t other sets. In this case y contributes nothing to the t-value of $\mathcal{W}$. Our main theorem shows that this intuition is correct.

Theorem 1.2. *In a set system $\mathcal{S}$ with $\bigcup \mathcal{S} \subseteq P$ and $|P| \ge |\mathcal{S}|$, **F** has a winning strategy against $\mathbf{B}_t$ if and only if $\left|\bigcup \mathcal{W}\right| \ge |\mathcal{W}| - \nu_t(\mathcal{W})$ for each $\mathcal{W} \subseteq \mathcal{S}$.*

We can recover Hall's Theorem from the case $t = |\mathcal{S}| - 1$; that is, $\mathbf{B}_t$ can remove all y's in $\mathcal{S}$ rendering **F**'s move equivalent to swapping the names of x and y, that is, rendering it useless. In Section 3 we show that Vizing's Theorem on edge-coloring is a quick corollary of this result. In fact, the strategy employed by **F** is based, in part, on the proofs of Vizing's Theorem by Ehrenfeucht, Faber, and Kierstead [2] and by Schrijver [5]. For a graph G, let $\chi'(G)$ be the edge-chromatic number of G and let $\Delta(G)$ be the maximum degree of G.

Corollary 1.3 (*Vizing [7]*)**.** *If G is a simple graph, then $\chi'(G) \le \Delta(G) + 1$.*

There is a "multiplicity" version of Hall's Theorem in which the representatives sought for the sets in the family are disjoint subsets of specified sizes. When each set S is asked to have $\eta(S)$ representatives in the "η-transversal", the desired subsets can be found by making $\eta(S)$ copies of each set S and applying Hall's Theorem. In Sections 4 and 5 we generalize this folklore extension of Hall's Theorem and use the generalization to give a non-standard proof of the following result from which classical edge-coloring results and various "adjacency lemmas" follow (see [6] for the standard proof and how these consequences are derived). Let xy be an edge in a multigraph G. We denote the multiplicity of xy by $\mu(xy)$. Additionally, xy is *critical* if $\chi'(G - xy) < \chi'(G)$.

Corollary 1.4. *Let G be a multigraph satisfying $\chi'(G) \ge \Delta(G) + 1$. For each critical edge xy in G, there exists $X \subseteq N(x)$ with $y \in X$ and $|X| \ge 2$ such that*

$$\sum_{v \in X} \big(d(v) + \mu(xv) + 1 - \chi'(G)\big) \ge 2.$$

2. The proof

Proof of Theorem 1.2. First we prove necessity of the condition. Suppose we have $\mathcal{W} \subseteq \mathcal{S}$ with $\left|\bigcup \mathcal{W}\right| < |\mathcal{W}| - \nu_t(\mathcal{W})$. We show that no matter what moves **F** makes, $\mathbf{B}_t$ can maintain this invariant. We then always have $\left|\bigcup \mathcal{W}\right| < |\mathcal{W}|$ and hence $\mathcal{W}$ can never have a transversal.

Suppose **F** modifies $S \in \mathcal{S}$ by inserting x and removing y to get S'. If $S \notin \mathcal{W}$, then $\mathbf{B}_t$ does not need to do anything, so we may assume $S \in \mathcal{W}$. Put $\mathcal{W}' = \mathcal{W} \cup \{S'\} \setminus \{S\}$.

If $d_{\mathcal{W}}(x) = 0$, then $\left|\bigcup \mathcal{W}'\right| = \left|\bigcup \mathcal{W}\right| + 1$. Now $\mathbf{B}_t$ swaps x in for y in $\min\{t, d_{\mathcal{W}'}(y)\}$ sets of $\mathcal{W}'$ to form $\mathcal{W}^*$. If $d_{\mathcal{W}'}(y) \le t$, then $d_{\mathcal{W}^*}(y) = 0$ and we have $\left|\bigcup \mathcal{W}^*\right| = \left|\bigcup \mathcal{W}\right|$; hence the invariant is maintained. Otherwise $\nu_t(\mathcal{W}^*) < \nu_t(\mathcal{W})$ because the degree of y has decreased by $t+1$, and again the invariant is maintained.

Hence we may assume $d_{\mathcal{W}}(x) > 0$. Now $\left|\bigcup \mathcal{W}'\right| \le \left|\bigcup \mathcal{W}\right|$. In order to have a chance to destroy the invariant, **F** must achieve $\nu_t(\mathcal{W}') > \nu_t(\mathcal{W})$. This requires $d_{\mathcal{W}'}(x) - 1$ to be a multiple of $t+1$ and $d_{\mathcal{W}'}(y)$ to not be a multiple of $t+1$; in particular, $d_{\mathcal{W}'}(y) \ne d_{\mathcal{W}'}(x) - 1$. If $d_{\mathcal{W}'}(y) < d_{\mathcal{W}'}(x) - 1$, then $\mathbf{B}_t$ swaps y in for x in one set in $\mathcal{W}' \setminus \{S'\}$. Doing so maintains the invariant, since now every element has the same degree in the new set system as in $\mathcal{W}$. Otherwise, $d_{\mathcal{W}'}(y) > d_{\mathcal{W}'}(x) - 1$ and $\mathbf{B}_t$ swaps x in for y in $\min\{t, d_{\mathcal{W}'}(y) + 1 - d_{\mathcal{W}'}(x)\}$ sets of $\mathcal{W}'$. This reduces the contribution from y without further increasing the contribution from x and thereby maintains the invariant.

Now we prove sufficiency. Suppose the condition is not sufficient for **F** to have a winning strategy. Among all counterexamples having the fewest sets, choose $\mathcal{S}$ to maximize $\left|\bigcup \mathcal{S}\right|$.

First, suppose $\left|\bigcup \mathcal{S}\right| \ge |\mathcal{S}|$. Let C be a minimal nonempty subset of $\bigcup \mathcal{S}$ such that $|\mathcal{W}_C| \le |C|$, where $\mathcal{W}_C = \{S \in \mathcal{S} \mid C \cap S \ne \emptyset\}$ (we can make this choice because $\bigcup \mathcal{S}$ is such a subset). Create a bipartite graph with parts C and $\mathcal{W}_C$ and an edge from $x \in C$ to $S \in \mathcal{W}_C$ if and only if $x \in S$. If $|C| = 1$, then we clearly have a matching of C into $\mathcal{W}_C$. Otherwise, by minimality of C, for every set D such that $\emptyset \ne D \subset C$ we have $|\mathcal{W}_D| > |D|$ and hence $|C| = |\mathcal{W}_C|$; now applying Hall's Theorem (for bipartite graphs) gives a matching of C into $\mathcal{W}_C$. This matching gives a transversal $f\colon \mathcal{W}_C \hookrightarrow \bigcup \mathcal{W}_C$ with image C. Put $\mathcal{S}' = \mathcal{S} \setminus \mathcal{W}_C$ and $P' = P \setminus C$. The hypotheses of the claim are satisfied by $\mathcal{S}'$ and P'. If **F** continues to play only using $\mathcal{S}'$ and $\mathcal{P}'$, then $\mathbf{B}_t$ cannot destroy the transversal of $\mathcal{W}_C$ that exists using elements of C, even though $\mathbf{B}_t$ may play on all of $\mathcal{S}$, because **F** will make no further move involving the elements in that transversal. Now minimality of $|\mathcal{S}|$ gives a contradiction.

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