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Unique square property, equitable partitions, and product-like graphs

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ABSTRACT

Equivalence relations on the edge set of a graph G that satisfy restrictive conditions on chordless squares play a crucial role in the theory of Cartesian graph products and graph bundles. We show here that such relations in a natural way induce equitable partitions on the vertex set of G, which in turn give rise to quotient graphs that can have a rich product structure even if G itself is prime.

1. Introduction

Sabidussi [20] and later Vizing [22] showed that every finite connected graph has a unique prime factorization w.r.t. the Cartesian product. This Cartesian product structure is naturally understood in terms of an equivalence relation σ on the edge set $E(G)$ that identifies the fibers as the connected components of the subgraphs of G that are induced by a single equivalence class of σ [20]. The first polynomial time algorithm to compute the factorization of an input graph [4] explicitly constructs σ starting from another, finer, relation δ. The product relation σ was later shown to be simply the convex hull $\mathfrak{C}(\delta)$ of the relation δ [17].

Graph bundles [18], the combinatorial analog of the topological notion of a fiber bundle [15], are a common generalization of both Cartesian products [10] and covering graphs [1]. A slight modification of the relation δ turns out to play a fundamental role for the characterization of graph bundles [24] and forms the basis of efficient algorithms to recognize Cartesian graph bundles [16,23,24]. Here we introduce a further generalization, termed USP-relations, that still retains the salient properties of δ, and hence can be viewed as a further relaxation of the product relation giving rise to a class of product-like graphs even more general than graph bundles. Generalizations of product relations provide an avenue to product-like graphs that

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is fundamentally different from the "approximate product graphs" studied in [12,13,11,14] in terms of coverings by small factorizable subgraphs. We thus strive to understand in more detail whether graphs that admit generalized product relations on their edge sets also have other properties akin to true product graphs.

The connected components of a given equivalence class of the product relation σ, i.e., the fibers of G w.r.t. to a given factor F, form a natural partition $\mathcal{P}_F$ of the vertex set of G. It is well known (see e.g. [10]) that G then has a representation as $G \cong (G/\mathcal{P}_F)\Box F$. This simple observation suggests considering quotient graphs of Cartesian products in a more systematic way.

Equitable partitions of graphs [5,6] were originally introduced as a means of simplifying the computation of graph spectra [21] and walks on graphs [8]. A series of recent results on so-called perfect state transfer revealed a close connection between equitable partitions of the vertex set of G, the corresponding quotient graphs, and the Cartesian product structure of G [2,7]. We therefore ask whether this connection persists in product-like graphs.

We show here that equivalence relations that are coarsenings of relations with the unique square property on the edge set $E(G)$, i.e., that are much more general than product relations, also induce equitable partitions on the vertex set $V(G)$. The quotient graphs w.r.t. these equitable partitions exhibit a natural, rich product structure even when G itself is prime.

2. Background and preliminaries

2.1. Basic definitions and notation

In the following we assume that G is a finite connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. A graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. H is an *induced subgraph* of G if $x, y \in V(H)$ and $[x, y] \in E(G)$ implies $[x, y] \in E(H)$. An induced cycle on four vertices is called a *chordless square*. We use square brackets $[x, y]$ to denote edges of undirected graphs and round brackets (x, y) for directed edges in digraphs.

Relations. We will consider equivalence relations R on E, i.e., $R \subseteq E \times E$ such that (i) $(e, e) \in R$, (ii) $(e, f) \in R$ implies $(f, e) \in R$ and (iii) $(e, f) \in R$ and $(f, g) \in R$ implies $(e, g) \in R$. The equivalence classes of R will be denoted by Greek letters, $\varphi \subseteq E$. We will furthermore write $\varphi \sqsubseteq R$ to mean that φ is an equivalence class of R.

A relation Q is finer than a relation R while the relation R is coarser than Q if $(e, f) \in Q$ implies $(e, f) \in R$, i.e., $Q \subseteq R$. In other words, for each class ϑ of R there is a collection $\{\chi \,|\, \chi \subseteq \vartheta\}$ of Q-classes, whose union equals ϑ. Equivalently, for all $\varphi \sqsubseteq Q$ and $\psi \sqsubseteq R$ we have either $\varphi \subseteq \psi$ or $\varphi \cap \psi = \emptyset$.

To make this paper easier to read, in the following we denote refinements of a given relation R by Q and coarse grainings of R by S, so that $Q \subseteq R \subseteq S$.

For a given equivalence class $\varphi \sqsubseteq R$ and a vertex $u \in V(G)$ we denote the set of neighbors of u that are incident to u via an edge in φ by $N_\varphi(u)$, i.e.,

$$N_\varphi(u) := \{v \in V(G) \mid [u, v] \in \varphi\}.$$

Equitable partitions. A partition $\mathcal{P}$ of the vertex set $V(G)$ of a graph G is *equitable* if, for all (not necessarily distinct) classes $A, B \in \mathcal{P}$ every vertex $x \in A$ has the same number

$$m_{AB} := |N_G(x) \cap B|$$

of neighbors in B. The $|\mathcal{P}| \times |\mathcal{P}|$ matrix $\mathbf{M} = \{m_{AB}\}$ indexed by the classes of $\mathcal{P}$ is known as the *partition degree matrix*.

Quotient graphs. Let G be a graph and $\mathcal{P}$ be a partition of $V(G)$. The (undirected) *quotient graph* $G/\mathcal{P}$ has as its vertex set $\mathcal{P}$, i.e., the classes of the partition. There is an edge $[A, B]$ for $A, B \in \mathcal{P}$ if and only if there are vertices $a \in A$ and $b \in B$ such that $[a, b] \in E(G)$. Note that there is a loop $[A, A]$ unless the class A of $\mathcal{P}$ is an independent set.

Weighted quotient graphs. Let G be a graph and let $\mathcal{P}$ be an equitable partition of $V(G)$ with partition degree matrix $\mathbf{M}$. The *directed weighted quotient graph* $\overrightarrow{G/\mathcal{P}}$ has vertex set $V(\overrightarrow{G/\mathcal{P}}) = \mathcal{P}$ and directed edges (A, B) from A to B with weight m_{AB} iff $m_{AB} \geq 1$. Note that $\overrightarrow{G/\mathcal{P}}$ has loops whenever $m_{AA} \geq 1$.

By construction, $m_{AB} \geq 1$ implies $m_{BA} \geq 1$. Hence $\overrightarrow{G/\mathcal{P}}$ has a well-defined underlying undirected and unweighted graph, which obviously coincides with $G/\mathcal{P}$. The underlying simple graph, obtained by also omitting the loops, will be denoted by $\mathcal{N}(\overrightarrow{G/\mathcal{P}}) = \mathcal{N}(G/\mathcal{P})$.

Cartesian graph product. The *Cartesian product* $G\Box H$ has vertex set $V(G\Box H) = V(G) \times V(H)$; two vertices (g_1, h_1), (g_2, h_2) are adjacent in $G\Box H$ if $[g_1, g_2] \in E(G)$ and $h_1 = h_2$, or $[h_1, h_2] \in E(H)$ and $g_1 = g_2$.

Cartesian products generalize in a natural way to directed edge-weighted graphs (with loops allowed). Their Cartesian product $G\Box H$ has the edge weights

$$m((g_1, h_1), (g_2, h_2)) = \begin{cases} m_G(g_1, g_2), & \text{iff } h_1 = h_2 \text{ and } g_1 \neq g_2 \\ m_H(h_1, h_2), & \text{iff } g_1 = g_2 \text{ and } h_1 \neq h_2 \\ m_G(g_1, g_2) + m_H(h_1, h_2), & \text{iff } g_1 = g_2 \text{ and } h_1 = h_2 \\ 0, & \text{otherwise} \end{cases} \quad (1)$$