Discrete Mathematics 313 (2013) 2423–2431

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Polychromatic 4-coloring of cubic even embeddings on the projective plane

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ARTICLE INFO

Article history:
Received 18 March 2012
Received in revised form 24 June 2013
Accepted 2 July 2013
Available online 27 July 2013

Keywords:
Polychromatic coloring
Even embedding
Projective plane
Möbius ladder
Eulerian triangulation

ABSTRACT

A *polychromatic k-coloring* of a map G on a surface is a k-coloring such that each face of G has all k colors on its boundary vertices. An *even embedding* G on a surface is a map of a simple graph on the surface such that each face of G is bounded by a cycle of even length. In this paper, we shall prove that a cubic even embedding G on the projective plane has a polychromatic proper 4-coloring if and only if G is not isomorphic to a Möbius ladder with an odd number of rungs. For proving the theorem, we establish a generating theorem for 3-connected Eulerian multi-triangulations on the projective plane.

1. Introduction

A *surface* $\mathbb{S}$ means a compact connected two-dimensional manifold without boundary. A simple closed curve ℓ on $\mathbb{S}$ is *contractible* if ℓ bounds a 2-cell on $\mathbb{S}$, and ℓ is *essential* otherwise. Moreover, ℓ is *one-sided* if a tubular neighborhood of ℓ is homeomorphic to a Möbius band, and *two-sided* otherwise. If $\mathbb{S} - \ell$ is disconnected, then ℓ is *separating*.

For a graph G, let $V(G)$ and $E(G)$ denote the sets of vertices and edges, respectively. A graph G is *cubic* if each vertex of G has degree exactly 3. A *cycle* is a closed walk with no repeated vertices, and a *k-cycle* is a cycle of length k. A cycle is *even* (resp., *odd*) if it is of even (resp., odd) length. A vertex of degree k is a *k-vertex*. A vertex v of G is a *cut vertex* if $G - v$ is disconnected. Moreover, a subset $S \subset V(G)$ with $|S| = k$ is a *k-cut* if $G - S$ is disconnected. (A subgraph H in G is *separating* if $G - V(H)$ is disconnected.) Similarly, a *cut edge* and a *k-edge cut* are defined for edges. Clearly, if a cubic graph G has a k-cut, then G also has a k-edge cut. A graph G is *k-connected* if $|V(G)| \geq k + 1$ and G has no l-cut for any $l < k$. On the other hand, G is *k-edge-connected* if G has no l-edge cut for any $l < k$.

Let G be a *map* on $\mathbb{S}$, that is, a fixed embedding of a simple graph on $\mathbb{S}$. For a vertex v of G, the *link* of v is the boundary walk of the union of the faces incident to v. A *triangulation* is a map with each face triangular. An *even embedding* is a map such that each face is bounded by a closed walk of even length. A *quadrangulation* is an embedding with each face quadrilateral. (We sometimes deal with graphs or maps allowed to have multiple edges but no loops, and in this case, we add the prefix "multi-" to those graphs and maps.) It is well-known that every even embedding on the sphere is bipartite, but this does not hold for any other surface. A graph (or map) is *Eulerian* if each vertex has even degree.

Let G be a map on a surface. A color-assignment $c : V(G) \to \{1, \dots, k\}$ is a *k-coloring* of G, and c is *proper* if for any $xy \in E(G)$, $c(x) \neq c(y)$. A k-coloring c (which is not necessarily proper) is *polychromatic* if every face of G receives all the k colors on its boundary vertices, and G is *k-polychromatic* if G admits a polychromatic k-coloring. In particular, G is *properly* k-polychromatic if G admits a polychromatic proper k-coloring. Observe that every map G is 1-polychromatic, and that if

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http://dx.doi.org/10.1016/j.disc.2013.07.005

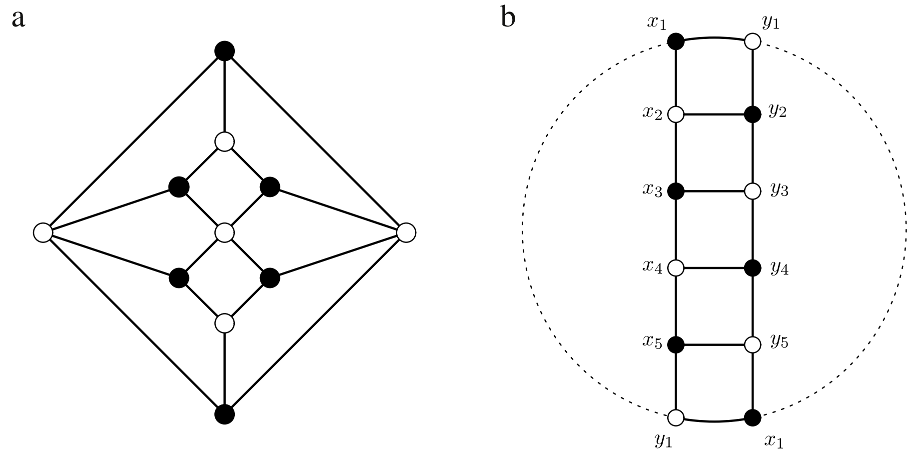

Fig. 1. Non-4-polychromatic non-cubic bipartite plane graph and odd Möbius ladder.

G is k-polychromatic for some $k \geq 2$, then G is also $(k-1)$-polychromatic. So there is a largest integer k such that G is k-polychromatic, which is the *polychromatic number* of G and denoted as $p(G)$.

The polychromatic coloring of plane graphs was defined by Horev and Krakovski; it was introduced in [1] in relation to the art gallery problem [5,6]. By definition, it is easy to see that for any map G on a surface, the polychromatic number $p(G)$ is not greater than the minimum face size of G, denoted as $g(G)$. Moreover, if a plane graph G has a spanning subgraph H, then $p(H) \geq p(G)$. It was proved in [1] that for any plane graph G, it holds that $p(G) \geq \lfloor \frac{3g(G)-5}{4} \rfloor$, which is tight up to an additive constant. Moreover, Horev and Krakovski [8] proved that every plane graph G with maximum degree at most 3 is (properly) 3-polychromatic if G is isomorphic to neither K_4 nor a graph obtained from K_4 by subdividing one edge by a single vertex.

For bipartite plane graphs, Horev et al. [7] proved the following:

Theorem 1 (*Horev et al. [7]*)**.** *Every cubic bipartite plane graph is properly* 4*-polychromatic.*

In Theorem 1, "4" is best possible since such a plane graph must have a quadrilateral face, by Euler's formula. Moreover, the result does not hold if we omit the cubicity from the assumption. (See Fig. 1(a), which shows a non-cubic bipartite plane graph with no polychromatic 4-coloring.)

In this paper, we consider an analogue of Theorem 1 for even embeddings on $\mathbb{P}$, where $\mathbb{P}$ stands for the projective plane throughout the paper. Note that an even embedding is always *simple*. A *Möbius ladder*, denoted as M_k, is an embedding of a $2k$-cycle $x_1 \cdots x_k y_1 \cdots y_k$ with k edges $x_1y_1, \ldots, x_ky_k$ on $\mathbb{P}$ such that the $2k$-cycle bounds a face, where each edge x_iy_i is a *rung*, and the $2k$-cycle is its *boundary*. See Fig. 1(b), in which we identify each antipodal pair of points of the dotted circle to get $\mathbb{P}$. A Möbius ladder M is *odd* if M has an odd number of rungs. Obviously, a Möbius ladder is bipartite if and only if it is odd.

Our main theorem is as follows:

Theorem 2. *Let G be a cubic even embedding on the projective plane. Then G is properly* 4*-polychromatic if and only if G is not isomorphic to an odd Möbius ladder.*

Like in Theorem 1, "4" is best possible by Euler's formula. In Theorem 2, if we omit the cubicity from the assumption, then we can find non-4-polychromatic even embeddings on $\mathbb{P}$. See Fig. 2, in which G_1 and G_2 are non-4-polychromatic even embeddings, with a 2-vertex and with a 4-vertex, respectively. On the other hand, if we omit the evenness of embeddings, then the pentagonal embedding G_3 of a Petersen graph shown in Fig. 2 is not 4-polychromatic. Hence the assumption for G in Theorem 2 cannot be weakened.

Recently, it has been proved in [12] that a quadrangulation G on a surface is properly 3-polychromatic if the dual 4-regular map of G has no essential "straight closed walk". (In a 4-regular map K, a "straight walk" W is a walk in K such that at each vertex v of W, W passes through e_i and e_{i+2} for some i, where e_1, e_2, e_3, e_4 are edges incident to v lying in this cyclic order.) In particular, this condition is also necessary for a quadrangulation on $\mathbb{P}$ to be properly 3-polychromatic. So it is an interesting problem to characterize properly 3-polychromatic quadrangulations on surfaces. Moreover, it will be interesting to consider which quadrangulations on surfaces have polychromatic 3-colorings which are not necessarily proper.

In order to prove Theorem 2, we establish a generating theorem for 3-connected Eulerian multi-triangulations on $\mathbb{P}$, whose dual maps are 3-connected cubic even embeddings with some properties. This has not yet been done in the literature, though the generating theorems for *simple* Eulerian triangulations are known for the sphere [2] and the projective plane [14].