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Note

On the relative distances of eleven points in the boundary of a plane convex body ☆

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A B S T R A C T

Let C be a plane convex body. The relative distance (or C-distance) of points $a, b \in R^2$ is defined by the ratio of the Euclidean length of the line-segment ab to half of the Euclidean length of a longest chord of C, parallel to ab. It was conjectured that there exists no plane convex body whose boundary contains eleven points at pairwise relative distances greater than $\frac{2}{3}$. We give an affirmative answer to this conjecture.

1. Introduction and notations

Let k (≥ 2) be an integer, to find k points on the sphere or in the ball of a Euclidean n-space E^n such that their pairwise distances are as large as possible is a long-standing problem in geometry. A generalization of this problem was presented by Doyle, Lagarias and Randall [4], and by Lassak [10]. Doyle et al. considered the points in the boundary of the unit ball C of a Minkowski plane [4]. Lassak gave a more general approach, that is, let C be an arbitrary plane convex body and the problem is to find configurations of points in the boundary of C, whose pairwise distances are large in the sense of the following notion of C-distance of points [10]. Some results concerning this kind of distance appeared in [1–3,5–15].

We use some definitions from [3]. For arbitrary different points $a, b \in E^2$, denote by ab the line-segment connecting the points a and b, by $|ab|$ the Euclidean length of ab, and by $\overline{ab}$ the straight line passing through a and b. The C-*distance* $d_C(a, b)$ of points a, b is defined by the ratio of $|ab|$ to $\frac{1}{2}|a_1b_1|$, where a_1b_1 is a longest chord of C parallel to ab. If there is no confusion about C, we use the term *relative distance* of a and b. Observe that for arbitrary points $a, b \in E^2$ the C-distance of a and b is equal to their $[\frac{1}{2}(C - C)]$-distance.

Denote by $\mu_k(C)$ the greatest possible number d such that the boundary of C contains k points at pairwise C-distances at least d, and denote by $\mathcal{C}$ the family of plane convex bodies.

Let

$$\mu_k(\mathcal{C}) = \sup\{\mu_k(C) \mid C \in \mathcal{C}\}.$$

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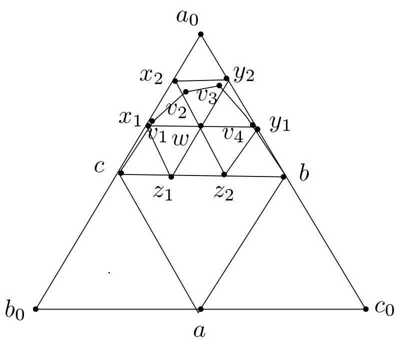

Fig. 1. Case 1.

Clearly, $\mu_k(C) \le 2$ for any convex body C, and $\mu_k(\mathcal{C}) = 2$ for $k = 2, 3, 4$. It is known that $\mu_5(\mathcal{C}) = \sqrt{5} - 1$ [3], $\mu_6(\mathcal{C}) = 8 - 4\sqrt{3}$, $\mu_7(\mathcal{C}) = 1$ [7], $\mu_8(\mathcal{C}) = 1$ [13], and $\mu_9(\mathcal{C}) \ge \sqrt{3} - 1$, $\mu_{10}(\mathcal{C}) > 2/3$ (see [14]).

Lángi [8, Chapter 3] conjectured that *there exists no plane convex body whose boundary contains nine points at pairwise relative distances greater than* $4 \sin \frac{\pi}{18}$, *and that there exists no plane convex body whose boundary contains ten (or eleven) points at pairwise relative distances greater than* $\frac{2}{3}$. *Since these numbers are attained for some particular bodies, he formulated the conjectures that* $\mu_9(\mathcal{C}) = 4 \sin \frac{\pi}{18}$ *and* $\mu_{10}(\mathcal{C}) = \mu_{11}(\mathcal{C}) = \frac{2}{3}$.

In 2009, Lan and Su showed that $\mu_9(\mathcal{C}) \ge 6 - 2\sqrt{7}$, which is over $4 \sin \frac{\pi}{18}$. Thus the above conjecture about nine points fails [6]. In 2012, Su et al. improved the lower bound for $\mu_9(\mathcal{C})$ to $\sqrt{3} - 1$ and also disproved the above conjecture about ten points by showing that $\mu_{10}(\mathcal{C}) > \frac{2}{3}$ (see [14]). In this paper, we confirm the conjecture about eleven points.

2. The main results

In order to formulate our result about the relative distances of eleven points in the boundary of a plane convex body we need the following observation. For arbitrary points $a, b \in E^n$ and for arbitrary convex bodies $C_1 \subset C_2$, it is clear that $d_{C_2}(a, b) \le d_{C_1}(a, b)$. Therefore, to find the upper bound of the minimum pairwise relative distances of k points in the boundary of an arbitrary plane convex body it suffices to examine convex k-gons inscribed in this body. Moreover, in order to determine $\mu_k(\mathcal{C})$, we only need to examine the relative distances of every two *consecutive* vertices of convex k-gons (see Lemma 3 together with the comment just after the proof of Lemma 6 and the last paragraph of p. 146 of [11]). If two lines $\overline{pq}$ and $\overline{rs}$ are parallel, we write $\overline{pq} \parallel \overline{rs}$.

Theorem. *Every convex hendecagon has two consecutive vertices at relative distance at most* $\frac{2}{3}$.

Proof. Denote by E the given convex hendecagon. Let $T = abc$ be a triangle formed by three vertices of E with the maximal possible area. For every non-degenerate affine transformation ϕ and for arbitrary points $p, q \in C$ we know that $d_{\phi(C)}(\phi(p), \phi(q)) = d_C(p, q)$. Thus we suppose, without loss of generality, that T is a regular triangle. Let T_0 be the image of T under the homothety of ratio -2 and with the homothety center in the center of gravity of T. Denote by a_0, b_0, c_0 the vertices of T_0 such that a_0, b_0, c_0 are the images of points a, b, c, respectively. From the maximality of the area of T we conclude that all vertices of E belong to the triangle T_0. Denote by $V(E)$ the vertex set of E. Since $V(E) \setminus \{a, b, c\}$ has eight points, we consider the following two cases.

Case 1: at least four vertices from $V(E) \setminus \{a, b, c\}$ belong to one of the three triangles a_0bc, ab_0c, and abc_0.

Without loss of generality, we suppose that the triangle a_0bc contains at least four vertices from $V(E) \setminus \{a, b, c\}$. Denote by v_1 (resp. v_2, v_3, and v_4) the vertex adjacent to c (resp. v_1, v_4, and b) (which are in clockwise order). Let $x_1, x_2 \in a_0c$ and $|cx_1| = |x_1x_2| = |x_2a_0|$; $y_1, y_2 \in a_0b$ and $|by_1| = |y_1y_2| = |y_2a_0|$; $z_1, z_2 \in bc$ and $|cz_1| = |z_1z_2| = |z_2b|$ (see Fig. 1).

If one of the four relative distances $d_E(c, v_1)$, $d_E(v_1, v_2)$, $d_E(v_3, v_4)$, and $d_E(b, v_4)$ is at most $\frac{2}{3}$, then the result holds. Thus we may assume that all these four relative distances exceed $\frac{2}{3}$. By the convexity of E we know that the vertices v_2 and v_3 belong to the rhombus $a_0x_2wy_2$, where w is the midpoint of the segment x_1y_1. Let us take a Cartesian coordinate system such that a, b, and c are $(0, 0)$, $(1, \sqrt{3})$, and $(-1, \sqrt{3})$, respectively. The convexity of E implies that the slope of the line $\overline{v_2v_3}$ is between $-\sqrt{3}$ and $\sqrt{3}$, hence $d_E(v_2, v_3) \le \frac{2}{3}$. If v_2 and v_3 are consecutive vertices of E, then the result is true. Otherwise, one can take a vertex v of E between v_2 and v_3 and consider the side of E with endpoints v and v', then v and v' also belong to the rhombus $a_0x_2wy_2$. Again, by the convexity of E, we know that $d_E(v, v') \le \frac{2}{3}$.

Case 2: one of the three triangles a_0bc, ab_0c, abc_0 contains two vertices, and each of the other two triangles contains three vertices from $V(E) \setminus \{a, b, c\}$. We may assume that the triangle a_0bc contains two vertices from $V(E) \setminus \{a, b, c\}$.

Suppose that $c, x, y, z, a, u, v, w, b$ are consecutive vertices of E in counterclockwise order. If one of the inequalities $d_E(c, x) \le \frac{2}{3}$, $d_E(x, y) \le \frac{2}{3}$, $d_E(y, z) \le \frac{2}{3}$, and $d_E(z, a) \le \frac{2}{3}$ holds, then we are done. Consider the opposite case. We may also suppose that $d_E(a, u) > \frac{2}{3}$ and $d_E(w, b) > \frac{2}{3}$. Denote by f the intersection point of the lines $\overline{xy}$ and $\overline{az}$.