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Finite vertex-biprimitive edge-transitive tetravalent graphs[☆]

Cai Heng Li[a], Hua Zhang[b,a,*]

[a] School of Mathematics and Statistics, The University of Western Australia, Crawley, WA 6009, Australia

[b] School of Mathematics, Yunnan Normal University, Kunming 650091, PR China

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ABSTRACT

Ivanov and Iofinova classified vertex-biprimitive edge-transitive cubic graphs in 1985. As a natural generalization of Ivanov and Iofinova's work, in this paper we present a classification of tetravalent graphs which are G-vertex-biprimitive and G-edge-transitive for some automorphism group G of the graphs.

1. Introduction

In a highly cited article [8], Ivanov and Iofinova classified vertex-biprimitive edge-transitive cubic graphs, based on the classification of amalgams of edge-transitive cubic graphs obtained by Goldschmidt [7]. As a natural generalization of Ivanov and Iofinova's work, in this paper we classify vertex-biprimitive edge-transitive tetravalent graphs. Recall that a bipartite graph $\Gamma = (V, E)$ with two parts U and W is called *G-biprimitive* for some $G \le \mathrm{Aut}\,\Gamma$ if G acts primitively on both U and W, and is called *G-edge-transitive* if G is transitive on the edge set E. An *arc* of a graph is an ordered pair of adjacent vertices, and a graph Γ is called *G-arc-transitive* if $G \le \mathrm{Aut}\,\Gamma$ is transitive on the set of arcs.

The result of this paper is the following theorem.

Theorem 1.1. *Let Γ be a finite connected bipartite graph of valency 4. Assume that Γ is G-biprimitive and G-edge-transitive, where $G \le \mathrm{Aut}\,\Gamma$. Assume further that G is intransitive on V. Then for an edge $\{\alpha, \beta\}$, one of the following holds:*

(i) $\Gamma = \mathrm{K}_{4,4}$, *and* $\mathbb{Z}_2^2 : \mathbb{Z}_3 \le G < \mathrm{Aut}\,\Gamma = \mathrm{S}_4 \wr \mathbb{Z}_2$.

(ii) $|V| = 2p, 2p^2$, *or* $2p^3$, *with p a prime, and one of the following holds, respectively,*
$G = \mathbb{Z}_p : \mathbb{Z}_4$ *with* $p > 5$ *and* $4 \mid (p-1)$, *and* $\mathrm{Aut}\,\Gamma = G \times \mathbb{Z}_2$.
$G = \mathbb{Z}_p^2 : \mathbb{Z}_4$ *with* $4 \nmid (p-1)$, *or* $\mathbb{Z}_p^2 : \mathrm{D}_8$, *and* $\mathrm{Aut}\,\Gamma = (\mathbb{Z}_p^2 : \mathrm{D}_8) \times \mathbb{Z}_2$.
$G = \mathbb{Z}_p^3 : \mathrm{A}_4$ *or* $\mathbb{Z}_p^3 : \mathrm{S}_4$, *and* $\mathrm{Aut}\,\Gamma = (\mathbb{Z}_p^3 : \mathrm{S}_4) \times \mathbb{Z}_2$.

(iii) *Γ is the standard double cover of a vertex-primitive arc-transitive tetravalent graph.*

(iv) $G = \mathrm{PSL}_2(p)$, *where p is a prime with* $p \equiv \pm 13, \pm 37 \pmod{40}$, $G_\alpha \cong G_\beta \cong \mathrm{A}_4$, *and Γ is one of* $\frac{p+\varepsilon}{6}$ *non-isomorphic graphs, where* $\varepsilon = 1$ *or* -1 *such that 3 divides* $p + \varepsilon$.

(v) $G = \mathrm{PSL}_2(3^f)$ *with* $f \ge 3$ *prime*, $G_\alpha \cong G_\beta \cong \mathrm{A}_4$, *and Γ is one of* $\frac{3^{f-1}-1}{2}$ *non-isomorphic graphs.*

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* Corresponding author at: School of Mathematics, Yunnan Normal University, Kunming 650091, PR China.
E-mail addresses: cai.heng.li@uwa.edu.au (C.H. Li), Zhdahua@gmail.com, zhanghua@maths.uwa.edu.au (H. Zhang).

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Table A
Amalgams associated with almost simple groups.

K	K_α	K_β	Aut Γ	Comments
$PGL_2(11)$	D_{24}	S_4	$PGL_2(11)$	
$PSL_2(13)$	D_{12}	A_4	$PGL_2(13)$	
$PSL_2(23)$	D_{24}	S_4	$PSL_2(23)$	
$PSL_2(p)$	S_4	S_4	$PGL_2(p)$	$p \equiv \pm 1 \pmod 8$
$PSL_3(3)$	S_4	S_4	$PSL_3(3).2$	
$PSL_3(3)$	$3^2:2S_4$	$3^2:2S_4$	$PSL_3(3).2$	Projective plane
$PGL_3(7)$	$3^2:2A_4$	$6^2:S_3$	$PGL_3(7).2$	
$PGU_3(5)$	$3^2:2A_4$	$6^2:S_3$	$PGU_3(5).2$	
$PSp_4(3)$	$[3^3]:S_4$	$[3^3]:2A_4$	$PSp_4(3).2$	
$G_2(3)$	$(3_+^{1+2}\times 3^2):2S_4$	$(3_+^{1+2}\times 3^2):2S_4$	$G_2(3).2$	Generalized hexagon
M_{12}	$3^2:2S_4$	$3^2:2S_4$	$M_{12}.2$	Weiss graph
Th	$[3^9].2S_4$	$3^2.[3^7].2S_4$	Th	Discovered in [3]

(vi) *G is an almost simple group, there exists a normal subgroup $K \lhd G$ such that K, K_α, K_β and* Aut Γ *lie in Table A, and for each case there is a unique graph.*

Moreover, if Γ is not arc-transitive, then G is one of the following groups: $PSL_2(p)$ *as in part* (iv), $PSL_2(3^f)$ *as in part* (v), *or* $PGL_2(11)$, $PSL_2(13)$, $PSL_2(23)$, $PGL_3(7)$, $PGU_3(5)$, $PSp_4(3)$, *or* Th, *as in Table A.*

Remarks on Theorem 1.1:

(1) Unlike the cubic case, amalgams for tetravalent graphs are not known yet. The proof of Theorem 1.1 depends on the classification of primitive permutation groups which have soluble stabilizers, as given in [10].
(2) Finite vertex-primitive arc-transitive tetravalent graphs are classified in [9], and so graphs in part (iii) of Theorem 1.1 are known.
(3) All the graphs in (iv) are the standard double covers of certain undirected graphs or digraphs. Some of the graphs in (iv) are the standard double covers of graphs in (iii), while others are not, depending on the choice of G_β (see Lemma 5.7 for details). The graphs in (v) are the standard double covers of certain digraphs.
(4) To our best knowledge, the $PGL_3(7)$-graph and $PGU_3(5)$-graph are new examples, whereas all other graphs appearing in Theorem 1.1 have been known.
(5) Some graphs in Theorem 1.1 are actually vertex-transitive, such as some of the graphs in part (iii), and the graphs associated with the groups and stabilizers of the fourth row and the fifth row in Table A. The reader should keep in mind that by "vertex-biprimitive" we mean Aut Γ-vertex-intransitive. In this paper we only present G-vertex-biprimitive graphs because in some cases (for example in part (iv)) the vertex transitivity depends on the choice of G_β.

The layout of this paper is as follows. In Section 2 we present some of the graphs appearing in Theorem 1.1. In Section 3 we give some properties about the vertex stabilizers of edge-transitive tetravalent graphs. Then in Section 4, we give a reduction of the proof of Theorem 1.1 to the case of almost simple groups. Finally in Section 5, we deal with the groups of almost simple type, and in Section 6, we complete the proof of the main theorem.

2. Examples

In this section, we construct and study the graphs appearing in Theorem 1.1.

Let $\Gamma = (V, E)$ be a connected bipartite G-edge-transitive regular graph, say of valency k, and let $\{\alpha, \beta\}$ be an edge. Then the edge stabilizer $G_{\alpha\beta}$ coincides with $G_\alpha \cap G_\beta$, G_α is transitive on $\Gamma(\alpha)$ with stabilizer $(G_\alpha)_\beta = G_{\alpha\beta}$, and G_β is transitive on $\Gamma(\beta)$ with stabilizer $(G_\beta)_\alpha = G_{\alpha\beta}$. Thus $|G_\alpha : G_{\alpha\beta}| = |G_\beta : G_{\alpha\beta}| = k$. Since Γ is connected, it follows that $\langle G_\alpha, G_\beta\rangle = G$.

Conversely, for a group G and subgroups $L, R < G$ such that $L \cap R$ is *core-free*, one can define a G-edge-transitive graph with vertex set $V = [G : L] \cup [G : R]$, the disjoint union of the sets of the right cosets of L in G and R in G, such that

$$Lx \sim Ry \iff Ry \sim Lx \iff xy^{-1} \in LR.$$

This graph is called a *coset graph*, and is denoted by $\mathrm{Cos}(G, L, R)$. This coset graph is G-vertex-intransitive, and is not necessarily regular, the valencies of which are $|L : L \cap R|$ and $|R : L \cap R|$. A detailed study of such graphs can be found in [6].

First we list the following three results for later use, the proof of Lemma 2.1 is simple and can be seen in [6], while Lemma 2.3 is clear.

Lemma 2.1. *Let $\Gamma = (V, E)$ be a graph and let $G \le$ Aut Γ be transitive on E and intransitive on V. Then for an edge $\{\alpha, \beta\}$, Γ is isomorphic to* $\mathrm{Cos}(G, G_\alpha, G_\beta)$.

Lemma 2.2. *The coset graphs* $\mathrm{Cos}(G, L_1, R_1) = \mathrm{Cos}(G, L_2, R_2)$ *if and only if $L_2 = L_1^g$ and $R_2 = R_1^g$ for some $g \in G$.*