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Circuit extension and circuit double cover of graphs

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ABSTRACT

Let G be a cubic graph and C be a circuit. An extension of C is a circuit D such that $V(C) \subseteq V(D)$ and $E(C) \neq E(D)$. The study of circuit extension is motivated by the circuit double cover conjecture. It is proved by Fleischner (1990) that a circuit C is extendable if C has only one non-trivial Tutte bridge. It is further improved by Chan, Chudnovsky and Seymour (2009) that a circuit is extendable if it has only one odd Tutte bridge. Those earlier results are improved in this paper that C is extendable if all odd Tutte bridges of C are sequentially lined up along C. It was proved that if every circuit is extendable for every bridgeless cubic graph, then the circuit double cover conjecture is true (Kahn, Robertson, Seymour 1987). Although graphs with stable circuits have been discovered by Fleischner (1994) and Kochol (2001), variations of this approach remain one of most promising approaches to the circuit double cover conjecture. Following some early investigation of Seymour and Fleischner, we further study the relation between circuit extension and circuit double cover conjecture, and propose a new approach to the conjecture. This new approach is verified for some graphs with stable circuits constructed by Fleischner and Kochol.

1. Introduction

Let G be a bridgeless graph. A subgraph of G is *even* if every vertex is of even degree. A *circuit* of G is a connected 2-regular graph. The following is the well-known Circuit Double Cover Conjecture.

Conjecture 1.1 ([19,23,26,29]). *Every bridgeless graph G has a family of circuits that covers every edge precisely twice.*

Circuit Double Conjecture has been verified for K_5-minor-free graphs, Petersen-minor-free graphs [1,2] and graphs with specific structures such as Hamiltonian path [27], small oddness [18,16,17] and spanning subgraphs [13–15,30]. It suffices to show that the Circuit Double Cover Conjecture holds for cubic graphs [20]. The Circuit Double Cover Conjecture is strengthened to the Strong Circuit Double Cover Conjecture as follows.

Conjecture 1.2 (*Seymour, See [7, p. 237], [8], Also See [13]*). *Let G be a bridgeless cubic graph and C be a circuit of G. Then G has a circuit double cover which contains C.*

The Strong Circuit Double Cover Conjecture is related to Sabidussi's Compatibility Conjecture which asserts that if T is a Eulerian trail of a Eulerian graph G of minimum degree at least 4, there exists a circuit decomposition $\mathscr{D}$ of G such that no transition of T is contained in any element of $\mathscr{D}$. A circuit C of a graph G is a *dominating circuit* if $G - V(C)$ has no edges. Sabidussi's Compatibility Conjecture is equivalent to the following circuit cover version.

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Conjecture 1.3 (*Sabidussi and Fleischner [9], and Conjecture 2.4 in [3, p. 462]*). *Let G be a bridgeless cubic graph and C be a dominating circuit of G. Then G has a circuit double cover which contains C.*

There are few partial results known for Conjectures 1.2 and 1.3. It is well-known that 3-edge-colorable cubic graphs satisfy Conjectures 1.2 and 1.3. The following result is obtained by Fleischner.

Theorem 1.4 (*Fleischner [10], Also See [12]*). *Let G be a cubic graph with a circuit C such that $G - V(C)$ has only one vertex. Then G has a circuit double cover containing C.*

One way to attack these conjectures is circuit extension. This idea was first proposed by Seymour (see [11,22]). Given a circuit C of a bridgeless cubic graph G, a circuit D is called an *extension* of C if $V(C) \subseteq V(D)$ and $E(D) \neq E(C)$. If C has an extension in G, the pair (G, C) is *extendable*. The following is a problem proposed by Seymour.

Problem 1.5. Let G be a bridgeless cubic graph and C be a circuit. Is (G, C) extendable?

If the answer to Problem 1.5 is yes, then Conjectures 1.1–1.3 will follow (Proposition 6.1.4 in [31, p. 67]). However, Fleischner [11] constructed a counterexample to Problem 1.5 and answered Seymour's problem negatively. After that, Kochol [22] constructed an infinite family of cyclic 4-edge-connected cubic graphs G with circuits C such that (G, C) is not extendable. But it is still interesting to ask which cubic graphs have the circuit extension property.

Let G be a cubic graph and C be a circuit of G. Each component B of $G - E(C)$ is called a *Tutte-bridge* of C. A vertex in $B \cap C$ is called an *attachment* of B on C. A chord of C is a trivial Tutte-bridge. The *order* of B is the number of vertices in $B - V(C)$. A Tutte-bridge is *odd* if its order is odd.

Theorem 1.6 (*Fleischner [10]*). *Let G be a bridgeless cubic graph and C be a circuit of G. Then the circuit C is extendable if C has only one non-trivial Tutte-bridge.*

Theorem 1.7 (*Chan, Chudnovsky and Seymour [4]*). *Let G be a bridgeless cubic graph and C be a circuit of G. Then the circuit C is extendable if C has only one odd Tutte-bridge.*

In this paper, the above results are further strengthened in Theorem 1.9.

Definition 1.8. Let G be a bridgeless cubic graph and C be a circuit of G. All odd Tutte-bridges of C are *sequentially lined up along* C if each odd Tutte-bridge Q_i $(1 \le i \le t)$ of C has an attachment v_i such that $v_i v_{i+1} \in E(C)$ for $1 \le i \le t-1$.

Theorem 1.9. *Let G be a bridgeless cubic graph and C be a circuit of G. The circuit C is extendable if all odd Tutte-bridges of C are sequentially lined up along C.*

2. Proof of the main theorem

Let G be a cubic graph and M be a subset of $E(G)$. Let $G - M$ be the subgraph of G obtained from G by deleting all edges in M. The *suppressed graph* $\overline{G - M}$ is a graph obtained from $G - M$ by suppressing all vertices of degree two. If M is a matching, $\overline{G - M}$ is a cubic graph. If M has only one edge e, we use $G - e$ and $\overline{G - e}$ instead.

The following theorem will be used in the proof of the main theorem.

Theorem 2.1 (*Smith's Theorem, [28]*). *Let G be a cubic graph. Then every edge of G is contained in an even number of Hamiltonian circuits.*

Now, we are ready to prove the main theorem.

Proof of Theorem 1.9. Suppose that (G, C) is a minimum counterexample with $|E(G)|$ as small as possible. Let $T_1, \ldots, T_k$ be all odd Tutte-bridges of C such that each T_i has an attachment v_i and $v_i v_{i+1} \in E(C)$ for $i = 1, \ldots, k-1$.

(1) *For any edge $e \in E(G) \setminus E(C)$, $G - e$ is bridgeless.*

If not, assume that $e = uv \in E(G) \setminus E(C)$ is such that $G - e$ has a bridge $e' = u'v'$. Clearly, e is not a chord of C and neither is e'. Then $G - \{e, e'\}$ has two components Q and Q'. Without loss of generality, assume that $C \subseteq Q$ and $u, u' \in Q$. By parity, Q' is of even order. Let G' be the new cubic graph obtained from Q by adding a new edge uu'. Note that $T_1 \cap G', \ldots, T_k \cap G'$ are all odd Tutte-bridges of C in G', and $v_i v_{i+1}$ is still an edge of C in G'. Since $|E(G')| < |E(G)|$, (G', C) is extendable. Let D be an extension of C in G'. If D does not contain uu', then D is also an extension of C in G, a contradiction. If D contains uu', then $D - uu' + \{e, e'\} + P_{vv'}$ is an extension of C in G, where $P_{vv'}$ is a path of Q' joining v and v', a contradiction. The contradiction implies that $G - e$ is bridgeless.

(2) *Every non-trivial Tutte-bridge Q is acyclic.*

Suppose to the contrary that C has a Tutte-bridge Q which has a circuit. Let $e = uv$ be an edge on the circuit. Then the order of $\overline{Q - e}$ has the same parity as the order of Q. By (1), $\overline{G - e}$ is bridgeless. Note that the odd Tutte-bridges of C in $\overline{G - e}$ have the same property as the odd Tutte-bridges of C in G. Since $|E(\overline{G - e})| < |E(G)|$, C has an extension D in $\overline{G - e}$. Let D' be the corresponding circuit of D in G, which is an extension of C in G, a contradiction.

(3) *The circuit C is dominating.*

By (2), every non-trivial Tutte-bridge is a tree. We have to show that every non-trivial Tutte-bridge Q is $K_{1,3}$. Choose a leaf v of $Q - V(C)$ such that $v \notin N(v_i)$ for all $1 \le i \le t$, and let $e = uv$ be an edge of $Q - V(C)$. Then $Q - e$ has two