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The on-line degree Ramsey number of cycles

David Rolnick

Massachusetts Institute of Technology, Cambridge, MA, USA

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ABSTRACT

On-line Ramsey theory studies a graph-building game between two players. The player called Builder builds edges one at a time, and the player called Painter paints each new edge red or blue after it is built. The graph constructed is the *host graph*. Builder wins the game if the host graph at some point contains a monochromatic copy of a given *goal graph*. In the S_k*-game* variant of the typical game, the host graph is constrained to have maximum degree no greater than k. The *on-line degree Ramsey number* $\mathring{R}_\Delta(G)$ of a graph G is the minimum k such that Builder wins an S_k-game in which G is the goal graph. In this paper, we complete the investigation begun by Butterfield et al. into the on-line degree Ramsey numbers of n-cycles. Namely, we show that $\mathring{R}_\Delta(C_n) = 4$ for $n \geq 3$.

1. Introduction

The quintessential problem of graph Ramsey theory involves finding a monochromatic copy of a graph G within a larger graph whose edges are colored either red or blue. Given graphs G and H, we say that H *arrows* G if every 2-coloring of H contains a monochromatic copy of G as a subgraph. Two basic parameters of Ramsey theory are

- The *Ramsey number* $R(G)$ is the minimum number of vertices among graphs H that arrow G.
- The *size Ramsey number* $\hat{R}(G)$ is the minimum number of edges among graphs that arrow G.

These parameters may be defined similarly for s-colorings, $s > 2$. For the general case of s colors, Ramsey's Theorem states that, given any G, there exists N such that the complete graph K_n arrows G for $n \geq N$.

The numbers called *on-line* Ramsey numbers, introduced by Grytczuk, Haluszczak, and Kierstead [2], are based upon the following game: Two players, called Builder and Painter, generate a 2-colored graph H. Builder constructs edges one at a time, using some combination of existing vertices and new vertices. As each edge is built, Painter colors it either red or blue. Builder's goal is for the graph H to contain a monochromatic copy of some given graph G, and Painter's goal is to prevent this from happening. We will call the graph G the *goal graph* and the 2-colored graph H that is being built the *host graph*.

If Builder is allowed to build edges without constraint, then Ramsey's Theorem implies that by building a large complete graph she can force Painter to create a monochromatic copy of G. The next question is whether Builder can also win if she builds a sparse graph, instead of a complete one. We therefore restrict the game so that Builder is allowed to build only edges such that the host graph remains within a specified class of graphs $\mathcal{H}$. We call such a game an $\mathcal{H}$-game.

We will consider the case where $\mathcal{H}$ is the class S_k of graphs H such that $\Delta(H) \leq k$, where $\Delta(H)$ denotes the maximum degree of the graph H. The *on-line degree Ramsey number* $\mathring{R}_\Delta(G)$ is defined to be the minimum k such that Builder can win the S_k-game where G is the goal.

Butterfield et al. [1] studied $\mathring{R}_\Delta(G)$ extensively. Among their results were: (i) a complete classification of graphs G such that $\mathring{R}_\Delta(G) \leq 3$, (ii) a lower bound on $\mathring{R}_\Delta(G)$ for general graphs G, and (iii) an upper bound on $\mathring{R}_\Delta(T)$ for trees T. In [3], this upper bound is generalized to a game with more than two colors, and exact on-line degree Ramsey numbers are computed for certain graphs in the multicolor case.

E-mail address: drolnick@mit.edu.

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Much of [1] was devoted to an examination of the on-line degree Ramsey numbers of cycles. The following is a summary of their results in this area: The number $\mathring{R}_\Delta(C_n)$ equals either 4 or 5 for $n \geq 3$. Additionally, it equals 4 in the following cases: (i) n even, (ii) $n \geq 689$, (iii) $337 \leq n \leq 514$, (iv) $n = 3$.

We complete the investigation begun by [1] into the number $\mathring{R}_\Delta(C_n)$, showing that it equals 4 for $n \geq 3$. Our proof introduces useful new techniques in the study of on-line degree Ramsey numbers that may aid in the classification of general graphs G satisfying $\mathring{R}_\Delta(G) = 4$. In a separate paper [4], we classified all trees that satisfy this equation. We have also identified many other graphs, neither trees nor cycles, that satisfy it. However, the problem of full classification appears to us to be quite challenging.

Many other questions remain open relating to on-line degree Ramsey numbers, including the following, suggested in [1]: Can graphs G with maximum degree fixed at d have arbitrarily large on-line degree Ramsey number? One can also ask whether if it NP-hard to determine the winner of an S_k-game on a given goal graph. Further generalizations of known results in on-line Ramsey theory to a game of more colors could prove interesting.

2. Preliminary lemmas

A Painter is *consistent* if, whenever he is given a new edge e to be added to a host graph H, the color he assigns to e depends only on that component of $H \cup \{e\}$ containing e. Thus, for instance, a consistent Painter, when presented by Builder with an isolated edge, will always color it the same way, regardless of the other components of the host graph.

It was proved in [1] that, for any graph G and integer k, Builder can win the S_k-game with goal G if and only if Builder can win when Painter is constrained to play consistently. Hence, for the remainder of this paper, we will assume that Painter plays consistently.

Following [1], we define a *capacity function* on a graph G to be a function assigning a positive integer to each vertex. A *weighted graph* is a graph G with an associated capacity function. Given a weighted graph G and a non-weighted graph H, we say that H *contains* G if for some subgraph G' of H isomorphic to G, the degree in H of each vertex of G' is at most the capacity of the corresponding vertex of G.

In general, when we speak of "graphs" in this paper, we will mean weighted graphs. By a $(w_1, \ldots, w_n)$*-weighted* G, we will mean the graph G with capacities $w_1, \ldots, w_n$ at vertices $v_1, \ldots, v_n$, respectively, where $V(G) = \{v_1, \ldots, v_n\}$. If $w_1 = \ldots = w_n = w$, we will write simply w*-weighted* G. When referring to the claw $K_{1,3}$, we order the vertices with the degree-3 vertex first.

Our goal in this paper will be the proof that $\mathring{R}_\Delta(C_n) = 4$ for $n \geq 3$. Given the work of [1], it will suffice to prove $\mathring{R}_\Delta(C_n) = 4$ when n is odd and at least 5. We will divide this proof into three sections: (i) a short proof that holds for $n \geq 13$, (ii) a longer proof for the case $n = 5$, and (iii) a quite complicated proof that deals with the tricky cases of n equal to 7, 9, and 11.

Within an S_n-game with goal G, we say that Builder can *force* a 2-colored copy of a graph H if she can play such that Painter creates either a monochromatic copy of G (in which case Builder wins) or else a copy of H with the desired coloring. The following result is a variation on lemmas of [1].

Lemma 2.1. *Suppose that, in the S_4-game with goal C_n, Builder can force a copy of a 2-colored (weighted) graph H containing a vertex v of capacity at most 2.*

(a) *[Doubling Lemma] Builder can force a copy of the graph obtained by placing H alongside a copy H' of H and connecting the two copies by the edge vv', where the vertex v and its copy v' have their capacities increased by 2 each, no other capacities are changed, and the edge vv' gets whatever color Builder desires.*

(b) *[Extension Lemma] Builder can force a copy of the graph obtained from H by raising the capacity of v by 2 and adding a new vertex w of capacity 2, with the edge vw added in whatever color Builder desires.*

Proof. We first prove the Doubling Lemma. Suppose without loss of generality that Builder wants vv' to be constructed in red. Taking advantage of Painter's consistency, Builder forces n (2-colored) copies of H, then constructs an n-cycle on the copies of v. If all the edges of the cycle are painted blue, then Builder wins immediately because a monochromatic copy of C_n has been formed. Otherwise, Painter assigns red to some edge vv' and we have the desired H and H', with v and v' linked by a red edge and with the capacities at each of these vertices increased by 2.

We now prove the Extension Lemma. Suppose without loss of generality that Builder wants vw to be constructed in red. She then forces $n(n-1)/2$ copies of H and constructs an $n(n-1)$-cycle alternating between the copies of v and new vertices. If any edge of this cycle is red, then Builder has obtained the desired red edge vw. Otherwise, all edges are blue.

Builder now turns to the vertices in the $n(n-1)$-cycle that are not part of a copy of H and hence have capacity 2. She connects, for each i with $0 \leq i < n$, the $i(n-1)/2$th such vertex to the $(i+1)(n-1)/2$th. This presents Painter with an n-cycle, each of whose edges, if colored blue, completes a blue copy of C_n. Whatever Painter does, Builder wins. Note that throughout this construction every vertex in the host graph has degree at most 4, in keeping with the conditions of the S_4-game. □

3. The case $n \geq 13$

Theorem 3.1. $\mathring{R}_\Delta(C_n) \leq 4$ *when n is odd and at least* 13.