Discrete Mathematics 313 (2013) 2127–2138

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Generalized Bessel numbers and some combinatorial settings[☆]

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ARTICLE INFO

Article history:
Received 17 October 2012
Received in revised form 1 May 2013
Accepted 2 May 2013
Available online 24 May 2013

Keywords:
Stirling numbers
Bessel numbers
Partitions
G-Permutations
Telephone exchange
Semi-bipartite
Riordan matrices

ABSTRACT

Stirling numbers and Bessel numbers have a long history, and both have been generalized in a variety of directions. Here, we present a second level generalization that has both as special cases. This generalization often preserves the inverse relation between the first and second kind, and has simple combinatorial interpretations. We also frame the discussion in terms of the exponential Riordan group. Then the inverse relation is just the group inverse, and factoring inside the group leads to many results connecting the various Stirling and Bessel numbers.

1. Introduction

The *Bessel number* of the second kind, $B(n,k)$, is the number of ways in which an n-set X can be partitioned into k blocks of size one or two. These Bessel numbers have many properties similar to those of the Stirling numbers of the second kind. An explicit formula for $B(n,k)$ is well known as

$$B(n,k)=\begin{cases}\dfrac{n!}{2^{n-k}(n-k)!(2k-n)!} & \text{if } \left\lceil \dfrac{n}{2}\right\rceil \le k \le n,\\ 0 & \text{otherwise.}\end{cases} \tag{1}$$

If j is the number of singleton blocks in these set partitions, then the Bessel numbers can be displayed in matrix form by means of singleton blocks, the (n,j) entries being the Bessel numbers given by $B(n,\frac{n+j}{2})$ if $n+j$ is even and 0 if $n+j$ is odd:

$$\begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 1&0&1&0&0&0&0\\ 0&3&0&1&0&0&0\\ 3&0&6&0&1&0&0\\ 0&15&0&10&0&1&0\\ 15&0&45&0&15&0&1\\ &&&\cdots&&& \end{bmatrix}=\left\langle e^{z^2/2},z\right\rangle.$$

[☆] This work was supported by a National Research Foundation of Korea Grant funded by the Korean Government (NRF-2012-007648).

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http://dx.doi.org/10.1016/j.disc.2013.05.001

The $\langle , \rangle$ notation is the exponential Riordan group terminology,[1] which will be explained briefly at the end of this section. For an alternative approach, see [16].

If we think of a telephone call as a matching or a block of size 2, then we have the telephone exchange problem [10]. The total number of ways in which n subscribers to a telephone exchange can be connected is given by the row sums of the matrix, and the generating function is

$$1 + z + 2\frac{z^2}{2!} + 4\frac{z^3}{3!} + 10\frac{z^4}{4!} + 26\frac{z^5}{5!} + \cdots = e^{z+z^2/2}.$$

Here, the n subscribers could be vertices, and an edge connecting two vertices would represent a phone call. We would then be thinking about a labeled graph in which every vertex has degree 0 or 1. Then the connected subgraphs are either a single point or an edge. The connected part has the exponential generating function $z + z^2/2!$; see [1,3,4,14,15] for the exponential formula.

There are many possible variations. For instance, a cell phone could be turned off, so the blocks of size 1 now can be in two states. Then the generating function is $e^{2z+z^2/2!}$. Here is a different variation. We now also allow conference calls involving three parties. The connected part generating function is thus $z + z^2/2! + z^3/3!$, and the resulting generating function is $e^{z+z^2/2!+z^3/3!}$. It is natural to allow blocks (or conference calls) of every size giving the connected part generating function $z + z^2/2! + z^3/3! + \cdots = e^z - 1$. Now $(e^z - 1)^k/k!$ is the generating function when we want exactly k blocks and n subscribers (labeled vertices, distinguishable elements). If k is the number of blocks, we obtain the matrix $\langle 1, e^z - 1\rangle$, the entries being the Stirling numbers of the second kind, $S(n, k)$. Removing the restriction on block size gives the row sums, starting $1, 1, 2, 5, 15, 51, \ldots$, the Bell numbers with generating function e^{e^z-1}.

The dual is given by arranging each block in a circle (or in a line with the largest element at the head of the line). The connected part is

$$z + \frac{z^2}{2!} + 2!\frac{z^3}{3!} + 3!\frac{z^4}{4!} + \cdots = z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots = \ln\frac{1}{1-z}.$$

Now, the matrix is $\left\langle 1, \ln\frac{1}{1-z}\right\rangle$, the entries being the *unsigned* Stirling numbers of the first kind, $s(n, k)$. The row sums have the generating function

$$e^{\ln\frac{1}{1-z}} = \frac{1}{1-z} = 1 + z + z^2 + \cdots = \sum_{n\ge 0} n!\frac{z^n}{n!}.$$

This generating function summarizes the fact that any element of the symmetric group on n letters can be written uniquely as the product of disjoint cycles. The duality referred to earlier, after inserting minus signs on every other diagonal, can be expressed as

$$\langle 1, \ln(1+z)\rangle \left\langle 1, e^z - 1\right\rangle = \langle 1, z\rangle$$

in Riordan group terminology.

Bessel numbers and Stirling numbers have been generalized in a variety of directions. One direction in which Bessel numbers have been generalized is by permitting each block to be of size at most s for $s \ge 3$. We refer to these numbers as the *s-restricted numbers* of the second kind, $M_2^s(n, k)$ [6,7]. There is a second direction in which Stirling numbers of the second kind have been generalized as *r-Stirling numbers*, $S_r(n, k)$ [5]. Suppose that there are r distinguished elements in separate blocks or cycles. We may consider dividing up an army into squads, where the r lieutenants are in different squads. The Stirling numbers of the first kind $s_r(n, k)$ would generalize by having the r distinguished elements in separate cycles. For matchings, we might have r seeded players that we do not want to compete against each other.

In this paper, we present a second level generalization that has both the s-restricted numbers and the r-Stirling numbers as special cases. This generalization often preserves the inverse relation between the first kind and the second kind, and has simple combinatorial interpretations. We also frame the discussion in terms of the exponential Riordan group. Then the inverse relation is just the group inverse, and factoring inside the group leads to many results connecting the various Stirling, telephone exchange, and Bessel numbers.

Many combinatorial counting problems can be treated systematically using the Riordan group introduced by Shapiro, Getu, Woan, and Woodson [12]. Here, we use various elements from the exponential version of the Riordan group, and the generating functions are exponential generating functions, which we abbreviate as egfs.

An *exponential Riordan matrix* [2], also denoted as an *e-Riordan matrix*, is an infinite lower triangular matrix $L = [\ell_{n,k}]$ whose kth column has egf $g(z)f(z)^k/k!$, where $g(0) \ne 0, f(0) = 0$ and $f'(0) \ne 0$. Equivalently, $\ell_{n,k} = n![z^n]g(z)f(z)^k/k!$, where $[z^n]$ is the operator that extracts the coefficient of z^n. The matrix L is denoted by $\langle g(z), f(z)\rangle$. It is easy to show that, if we multiply $L = \langle g(z), f(z)\rangle$ by a column vector $\mathrm{h} = (h_0, h_1, \ldots)^T$ corresponding to the egf $h(z)$, then the resulting column

1 As is customary in this setting, we index our rows and columns starting with the index 0.