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Rotation number of a unimodular cycle: An elementary approach[☆]

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ABSTRACT

We give an elementary proof of a formula expressing the rotation number of a cyclic unimodular sequence $L = u_1u_2 \dots u_d$ of lattice vectors $u_i \in \mathbb{Z}^2$ in terms of arithmetically defined local quantities. The formula has been originally derived by A. Higashitani and M. Masuda [A. Higashitani, M. Masuda, Lattice multi-polygons, arXiv:1204.0088v2 [math.CO], [v2] Apr 2012; [v3] Dec 2012] with the aid of the Riemann–Roch formula applied in the context of toric topology. These authors also demonstrated that a generalized version of the 'Twelve-point theorem' and a generalized Pick's formula are among the consequences or relatives of their result. Our approach emphasizes the role of 'discrete curvature invariants' $\mu(a, b, c)$, where $\{a, b\}$ and $\{b, c\}$ are bases of $\mathbb{Z}^2$, as fundamental discrete invariants of *modular lattice geometry*.

1. Introduction

The following theorem of A. Higashitani and M. Masuda, proved in [4], is a close relative of the remarkable 'Twelve-point theorem' [10,1,3,5,8]. Like its predecessors, the 'Twelve-point theorem' and Pick's formula, it is an intriguing and easily formulated statement about a sequence of lattice vectors, their rotation (winding) number and the associated, arithmetically defined local quantities.

Theorem 1. *The rotation number* $\mathrm{Rot}(L)$ *of a cyclic unimodular sequence* $L = u_1u_2 \dots u_d$ *can be calculated as the weighted sum*

$$\mathrm{Rot}(L) = \frac{1}{12}\mu(L) + \frac{1}{4}\nu(L) = \frac{1}{12}\sum_{i=1}^{d} \mu(u_{i-1}, u_i, u_{i+1}) + \frac{1}{4}\sum_{i=1}^{d} \nu(u_i, u_{i+1}) \tag{1}$$

of locally defined quantities $\mu(L)$ *and* $\nu(L)$ *where* $\nu(u_i, u_{i+1}) := \det(u_i, u_{i+1}) \in \{-1, +1\}$ *and* $\mu(u_{i-1}, u_i, u_{i+1}) = a_i \in \mathbb{Z}$ *is the integer determined by the equation*

$$\det(u_{i-1}, u_i)u_{i-1} + \det(u_i, u_{i+1})u_{i+1} + a_iu_i = 0.$$

Theorem 1 may appear at first sight as quite elementary and not difficult to comprehend, however it has a deeper meaning and significance. Like its relative (and a consequence) the 'Twelve-point theorem', it is situated at the crossroads of several mathematical areas, illuminating and offering new perspectives on 'well understood' mathematical concepts.

The first proof [4] (see also [9, Section 5]) of the formula (1) was based on a Riemann–Roch type theorem (Noether formula) where the integers a_i appeared as the self-intersection numbers of the corresponding homology classes of the associated 'topological toric manifold'.

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Our first objective in this paper is to give a conceptual and elementary proof of Theorem 1 which is based on a systematic analysis of the invariant $\mu(a, b, c)$. The second objective is to pave the way for the hypothetical higher dimensional analogues of Theorem 1. For this reason the exposition emphasizes the study of invariants $\mu(a, b, c)$ and their higher dimensional versions $\mu_j(a, \mathfrak{b}, a')$ (Section 6) as 'discrete curvature invariants' situated within *unimodular lattice geometry*. This point of view is similar to the approach of O. Karpenkov to 'lattice trigonometry', as part of his investigation of 'lattice geometry invariants' [6,7].

A different, short and elementary computational proof of Theorem 1 was subsequently included in the new version of the paper [4]. Their Lemma 1.3. fits in nicely in our approach so we took the opportunity to shorten the original proof ([11], arXiv:1209.4981v1 [math.CO]), retaining its transparency and conceptuality.

We observe in passing (Section 5.1) that the proof of Poonen and Rodriguez-Villegas of the 'Twelve-point theorem' [10], based on the properties of the holomorphic function (modular form) $\Delta(z)$, can also be used as the basis of a proof of Theorem 1. The variety of proofs and methods applied are perhaps an indication that this result deserves a further exploration so the paper ends with some open questions.

2. Introductory definitions and remarks

2.1. Unimodular sequences

A sequence $L = u_1u_2 \ldots u_d$ of lattice vectors is called *unimodular* if $\{u_i, u_{i+1}\}$ is a basis of the lattice $\mathbb{Z}^2$ for each i or equivalently if $\det(u_i, u_{i+1}) \in \{-1, +1\}$ for each $i = 1, \ldots, d-1$. Geometrically this condition means that for each i the triangle Ou_iu_{i+1} does not contain lattice points aside from the vertices.

A unimodular sequence $L = u_1u_2 \ldots u_d$ is called *cyclic* if $\det(u_d, u_1) \in \{-1, +1\}$. A cyclic unimodular sequence L (of length d) naturally defines a d-periodic unimodular sequence $W = \ldots LLL \ldots = \ldots u_{-1}u_0u_1u_2 \ldots u_du_{d+1} \ldots$ where $u_i = u_j$ if $i \equiv j \pmod d$.

2.2. Local and global μ-invariants

The invariants $\mu(L)$ and $\nu(L)$ of a cyclic unimodular sequence $L = u_1 \ldots u_d$ are already introduced in Theorem 1 as the sums

$$\mu(L) = \sum_{i=1}^{d} \mu(u_{i-1}, u_i, u_{i+1}) \qquad \nu(L) = \sum_{i=1}^{d} \nu(u_i, u_{i+1}). \tag{2}$$

The μ-invariant of a unimodular sequence (u, v, w) is described as the unique integer $a = \mu(u, v, w)$ determined by the equation

$$\det(u, v)u + \det(v, w)w + av = 0. \tag{3}$$

Together with the associated ν-invariant $\nu(u, v) := \det(u, v)$ this is a basic discrete angle invariant of (planar) *modular lattice geometry*. Higher dimensional analogues of these invariants are introduced and discussed in Section 6 (see Definition 16).

A possible ambiguity arises if $L = u_1u_2u_3$ is a cyclic unimodular sequence. For this reason the term 'μ-invariant' is reserved for the number $\mu(u_1, u_2, u_3)$ (local μ-invariant) while $\mu(L) = \mu(u_1, u_2, u_3) + \mu(u_2, u_3, u_1) + \mu(u_3, u_1, u_2)$ is the corresponding global μ-invariant.

2.3. Rotation number

Let $P = P(a_1, \ldots, a_d)$ be a closed, oriented, polygonal curve in the plane with points a_i as vertices and $\overline{a_i, a_{i+1}} = [a_i, a_{i+1}]$ as oriented edges ($a_{d+1} := a_1$). If the origin O is not on P it has a *rotation number* (or winding number) defined by,

$$\mathrm{Rot}(P) = \frac{1}{2\pi} \sum_{i=1}^{d} \nu(a_i, a_{i+1}) \angle(a_iOa_{i+1}) \tag{4}$$

where $\nu(a_i, a_{i+1})$ is the sign of the determinant $\det(a_i, a_{i+1})$ and $\angle(a_iOa_{i+1})$ is the measure of the angle a_iOa_{i+1}.

Given a cyclic, unimodular sequence $L = u_1u_2 \ldots u_d$ the associated *closed unimodular polygon* is $P_L = P(u_1, u_2, \ldots, u_d)$. The *rotation number* $\mathrm{Rot}(L)$ of L is by definition the rotation number of the polygonal curve P_L.

It is well known that $\mathrm{Rot}(P)$ can be defined with the aid of elementary homology theory. We do not need this definition here but we shall occasionally use the term *unimodular cycle* $[L]$ to describe the collection $\{\overline{u_iu_{i+1}}\}_{i=1}^d$ of the oriented edges of P_L which may be written also as a formal sum,

$$[L] := \overrightarrow{u_1u_2} + \overrightarrow{u_2u_3} + \cdots + \overrightarrow{u_{d-1}u_d} + \overrightarrow{u_du_1}.$$

In this context the decomposition $[L] = [L_1] + [L_2]$, that appears in Section 4, simply indicates that $[L] = [L_1] \cup [L_2]$ is a union of sets with signed elements (multisets) where the elements with different sign, i.e. the edges with different orientation, are supposed to cancel out. The following proposition is, in light of the Eq. (4), an immediate consequence.