Discrete Mathematics 313 (2013) 2337–2347

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A Cheeger inequality of a distance regular graph using Green's function ☆

Gil Chun Kim *, Yoonjin Lee

Department of Mathematics, Ewha Womans University, Seoul, Republic of Korea

ARTICLE INFO

Article history:
Received 27 June 2012
Received in revised form 5 April 2013
Accepted 14 June 2013
Available online 3 July 2013

Keywords:
Green's function
Laplacian
P-polynomial scheme
Distance regular graph
Cheeger constant
Cheeger inequality

ABSTRACT

We give a Cheeger inequality of distance regular graphs in terms of the smallest positive eigenvalue of the Laplacian and a value α_d which is defined using q-numbers. We can approximate α_d with arbitrarily small positive error β. The method is to use a Green's function, which is the inverse of the β-Laplacian.

1. Introduction

Let $\Gamma = (V, E)$ be a graph with $|V| = v$ vertices. As given in [5], the *Cheeger constant* h_Γ of the graph Γ is defined by

$$h_\Gamma = \inf\left\{ \frac{|\partial S|}{\mathrm{vol}(S)} \mid S \subset V \text{ with } |S| \le \frac{|V|}{2} \right\},$$

where $\mathrm{vol}(S)$ is the sum of the degrees in S.

The Cheeger constant measures the difficulty of separating the graph into two large components by making a small edge-cut. The Cheeger constant of a connected graph is strictly positive. If the Cheeger constant is "small" but positive, then there are two large sets of vertices with "few" edges between them. On the other hand, if the Cheeger constant is "large", then any two sets of vertices must have many edges between those two subsets. We are interested in finding bounds on the Cheeger constants of graphs.

Let Γ be a connected graph. For a vertex $x \in V$, define $\Gamma_i(x)$ to be the set of vertices which are at distance precisely i from x $(0 \le i \le d)$, where $d = \max\{d(x, y) \mid x, y \in V\}$. A connected graph Γ with diameter d is called *distance regular* if there are integers b_i, c_{i+1} $(0 \le i \le d-1)$ such that for any two vertices x, y with $d(x, y) = i$, there are precisely c_i neighbors of y in $\Gamma_{i-1}(x)$ and b_i neighbors of y in $\Gamma_{i+1}(x)$.

The connectivity properties of distance regular graphs have been actively developed, for instance in [2,3,10]. From Propositions 1–3 we see that, for a distance regular graph, there are close connections between the Cheeger constant and vertex and edge connectivity.

☆ The second author was supported by Priority Research Centers Program through the NRF funded by the Ministry of Education, Science and Technology (2009-0093827) and by the NRF grant funded by the Korea government (MEST) (2012-0005432).

* Corresponding author.
E-mail addresses: higen@ewha.ac.kr (G.C. Kim), yoonjinl@ewha.ac.kr (Y. Lee).

0012-365X/$ – see front matter
http://dx.doi.org/10.1016/j.disc.2013.06.012

Proposition 1 ([2]). *Let Γ be a distance-regular graph with more than one vertex. Then its edge-connectivity equals its valency k, and the only disconnecting sets of k edges are the sets of edges incident with a single vertex.*

Proposition 2 ([3]). *Let Γ be a non-complete distance-regular graph of valency $k > 2$. Then the vertex-connectivity $\kappa(\Gamma)$ equals k, and the only disconnecting sets of vertices of size not more than k are the point neighborhoods.*

Proposition 3 ([10]). *Let $\Gamma = (V, E)$ be a simple graph with the vertex-connectivity $\kappa(\Gamma)$ and the edge-connectivity $\lambda(\Gamma)$. Then*

$$\frac{2\kappa(\Gamma)}{|V|} \leq \frac{2\lambda(\Gamma)}{|V|} \leq \inf \frac{|\partial S|}{|S|} \leq \kappa(\Gamma) \leq \lambda(\Gamma),$$

where S is a subset of V with $|S| \leq \frac{|V|}{2}$.

The eigenvalues of distance-regular graphs are involved with the intersection numbers [9,12]. The Cheeger constant has a relationship with the eigenvalues of the Laplacians of distance-regular graphs. In [8,11], a Cheeger constant is defined as $h_\Gamma = \inf_{S\subset V} \frac{|\partial S|}{\mathrm{vol}(S)}$ with $\mathrm{vol}(S) \leq \frac{1}{2}\mathrm{vol}(\Gamma)$, where vol$(S)$ is the sum of the degree in S. From Propositions 1–3, we obtain an inequality $\frac{2}{|V|} \leq h_\Gamma \leq 1$. We are thus interested in finding optimal bounds of h_Γ for distance regular graphs. The known inequalities of h_Γ with respect to λ_1 are as follows [8,11]:

$$h_\Gamma \leq \sqrt{2\lambda_1}, \qquad h_\Gamma \leq \sqrt{\lambda_1(2-\lambda_1)}, \tag{1}$$

where λ_1 is the smallest positive eigenvalue of the Laplacian $\mathcal{L}$.

In this paper, we find a Cheeger inequality of a distance regular graph. And this inequality can be expressed as some value α_d and λ_1 (Theorem 5), where α_d can be explicitly expressed as q-numbers. In general it is hard to compute the q-numbers, so we find an approximation of α_d with an arbitrarily small error (Theorem 10). As a major tool we use Green's function, which is defined as the inverse of β-Laplacian $\mathcal{L}_\beta$ [5,4,6]. We point out that our Cheeger inequality provides an upper bound of the Cheeger constant of the distance regular graph, which yields an improvement of the bound in Eq. (1).

We briefly explain the main results of the paper. Let Γ be a distance regular graph with diameter d and order v. Then, for any positive real value β, we obtain the components $r_i^{(\beta)}$ $(i = 0, 1, \ldots, d)$ and a Green's function $\mathcal{G}_\beta$. Throughout this paper, β is any positive real value. We consider a limit with respect to β, v, $r_i^{(\beta)}$ as follows:

$$\lim_{\beta\to 0^+} \frac{\beta^2 v r_i^{(\beta)}}{1-\beta v r_i^{(\beta)}} = \alpha_i, \quad (i = 0, 1, \ldots, d).$$

Then, we have $0 < \alpha_d < \cdots < \alpha_{e-1} < \alpha_e < \lambda_1$ for some e. Let h_Γ be the Cheeger constant of a distance regular graph. We prove the inequalities $\lambda_1 h_\Gamma < \alpha_d < \cdots < \lambda_1$ (Theorem 5) and hence the bound

$$h_\Gamma < \frac{\alpha_d}{\lambda_1}.$$

The values α_i's are expressed by the eigenvalues λ_j of the Laplacian and the q-numbers $q_j(i)$ of the P-polynomial scheme [1,7]. More precisely,

$$\alpha_i = \lim_{\beta\to 0^+} \frac{\beta^2 v r_i^{(\beta)}}{1-\beta v r_i^{(\beta)}} = \frac{1}{-q_1(i)\frac{1}{\lambda_1} - \cdots - q_d(i)\frac{1}{\lambda_d}}. \tag{2}$$

To evaluate α_d we need to determine the $q_j(d)$'s. But, the q-numbers are not easy to find in general. For this reason, we find an approximate value $\widetilde{\alpha_d}$ of α_d (Theorem 10). Moreover, we find an explicit expression of α_d by using the valencies k_j of the P-polynomial scheme and a certain nullspace $\mathcal{N}(L_{\mathrm{sub}}^{(\beta)})$.

In Section 2, we introduce basic facts of Cheeger inequalities and the P-polynomial scheme. In Section 3, we introduce Green's function $\mathcal{G}_\beta$ as the inverse of the β-Laplacian and give some properties of Green's function $\mathcal{G}_\beta$. In Section 4, we find the Cheeger inequality of the distance regular graph by using a Green's function $\mathcal{G}_\beta$, we obtain a Cheeger bound of the distance regular graphs. Finally, in Section 5, we find an approximate value $\widetilde{\alpha_d}$ of α_d with an arbitrarily small error. We also find an explicit expression of α_d by using the valencies k_j of the P-polynomial scheme and the basis of nullspace $\mathcal{N}(L_{\mathrm{sub}}^{(\beta)})$.

2. Preliminaries

In this section, we introduce the Cheeger constant, Cheeger inequality and basic facts on P-polynomial schemes.
Some definitions in graph theory.