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The acyclic and $\overrightarrow{C}_3$-free disconnection of tournaments

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ABSTRACT

The acyclic disconnection $\overrightarrow{\omega}(D)$ of a digraph D is defined as the maximum number of colors in a coloring of the vertices of D such that no cycle is properly colored (in a proper coloring, consecutive vertices of the directed cycle receive different colors). Similarly, the $\overrightarrow{C}_3$-free disconnection $\overrightarrow{\omega}_3(D)$ of D is the maximum number of colors in a coloring of the vertices of D such that no directed triangle is 3-colored. In this paper, we construct an infinite family $\mathfrak{V}_n$ of tournaments T_{8n+1} with $8n+1$ vertices ($n \in \mathbb{N}$) such that $\overrightarrow{\omega}_3(T_{8n+1}) = n+2$ and $\overrightarrow{\omega}(T_{8n+1}) = 2$. This family allows us to solve the following problem posed by V. Neumann-Lara [V. Neumann-Lara, The acyclic disconnection of a digraph, Discrete Math. 197/198 (1999) 617–632]: Are there tournaments T for which $\overrightarrow{\omega}(T) = 2$ and $\overrightarrow{\omega}_3(T)$ is arbitrarily large? The main result of the paper solves a generalization of the above problem: for positive integers r and s such that $2 \le r \le s$, there exists a tournament T such that $\overrightarrow{\omega}(T) = r$ and $\overrightarrow{\omega}_3(T) = s$.

1. Introduction

The *acyclic disconnection* $\overrightarrow{\omega}(D)$ of a digraph D is the maximum number of colors in a coloring of the vertices of D such that no cycle is properly colored (in a proper coloring, consecutive vertices of the directed cycle receive different colors). Let $\overrightarrow{C}_3$ denote a 3-cycle (a directed cycle of length 3). Similarly, the *$\overrightarrow{C}_3$-free disconnection* $\overrightarrow{\omega}_3(D)$ of D is the maximum number of colors in a coloring of the vertices of D such that no 3-cycle is 3-colored. A 3-colored 3-cycle is called a *heterochromatic* or *rainbow*. These notions were introduced in [6]. Actually, we can equivalently define $\overrightarrow{\omega}(D)$ (respectively, $\overrightarrow{\omega}_3(D)$) of a digraph D to be the maximum possible number of connected components (of the underlying graph) of a digraph obtained from D by deleting an acyclic (resp. $\overrightarrow{C}_3$-free) set of arcs, that is, a set of arcs not containing directed cycles (resp. 3-cycles), see Propositions 2.2 and 2.3 of [6]. We note that $\overrightarrow{\omega}(D) \le \overrightarrow{\omega}_3(D)$ for every digraph D; in particular, $2 \le \overrightarrow{\omega}(T) \le \overrightarrow{\omega}_3(T)$ for every tournament T.

These measures of the cyclic structure of a digraph have been studied in [3–6] for a variety of regular and circulant tournaments. In addition, the $\overrightarrow{C}_3$-free disconnection is closely related to the so-called "heterochromatic number" and, specifically, the "tightness" of 3-uniform hypergraphs. For more details on definitions and results, see [1,4,6].

An interesting example of a tournament T such that $\overrightarrow{\omega}(T) \ne \overrightarrow{\omega}_3(T)$ (in fact, $\overrightarrow{\omega}(T) = 2$ and $\overrightarrow{\omega}_3(T) = 3$) was constructed in [6], Example 4.2. In the same paper (Problem 6.1.3), V. Neumann-Lara conjectured that $\overrightarrow{\omega}(T) = \overrightarrow{\omega}_3(T)$ for every circulant or regular tournament T (the conjecture holds for circulant tournaments of prime order [4]). In this paper we solve the following

Problem 1.1 (*Problem 6.3.2 [6]*)**.** Are there tournaments T such that $\overrightarrow{\omega}(T) = 2$ and $\overrightarrow{\omega}_3(T)$ is arbitrarily large?

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The main result of this paper solves a generalization of the above problem: for positive integers r and s such that $2 \leq r \leq s$, there exists a tournament T such that $\overrightarrow{\omega}(T) = r$ and $\overrightarrow{\omega}_3(T) = s$.

2. Preliminaries

We set $[n] = \{1, \dots, n\}$. A digraph D has vertex set $V(D)$ and arc set $A(D)$. The *order* of D is $|V(D)|$ and we use $uv \in A(D)$ for an arc of D. If $S \subset V(D)$, then $D\langle S\rangle$ denotes the subgraph of D induced by S. An r-coloring of D is a surjective function $\varphi : V(D) \to \{c_i : i \in [r]\}$. Let $\overrightarrow{C}_n$ denote the directed cycle of length n. A directed cycle is *properly colored* under an r-coloring φ if any two consecutive vertices have distinct colors under φ. In particular, a properly colored 3-cycle is said to be *rainbow*. An *optimal coloring* of a digraph D with $\overrightarrow{\omega}(D) = r$ (resp. $\overrightarrow{\omega}_3(D) = r$) is an r-coloring that induces no properly colored cycle (resp. rainbow 3-cycle). An r-coloring of D is $\overrightarrow{C}_3$*-free* (resp. $\{\overrightarrow{C}_3, \overrightarrow{C}_4\}$*-free*) if D contains no rainbow 3-cycles (resp. rainbow 3-cycles or properly colored 4-cycles). As defined in [6], a digraph is $\overrightarrow{\omega}$*-keen* (resp. $\overrightarrow{\omega}_3$*-keen*) if there is an optimal proper coloring with exactly one singleton chromatic class (that is, a chromatic class with exactly one vertex). Note that no optimal coloring of $V(D)$ leaves more than one such class. For general concepts of digraphs see [2].

Let D and B be digraphs, and let $\{B_i\}_{i\in V(D)}$ be a family of isomorphic copies of B. The *composition* $D[B]$ is defined by

$$V(D[B]) = \bigcup_{i\in V(D)} V(B_i),$$

$$A(D[B]) = \bigcup_{i\in V(D)} A(B_i) \cup \{uv : u \in V(B_i), v \in V(B_j), ij \in A(D)\}.$$

Proposition 2.1 ([6] Proposition 3.6)**.** *Let D and B be digraphs,*

(i) *if D is $\overrightarrow{\omega}$-keen (resp. $\overrightarrow{\omega}_3$-keen), then $\overrightarrow{\omega}(D[B]) = \overrightarrow{\omega}(D) + \overrightarrow{\omega}(B) - 1$ (resp. $\overrightarrow{\omega}_3(D[B]) = \overrightarrow{\omega}_3(D) + \overrightarrow{\omega}_3(B) - 1$) and*
(ii) *if both digraphs D and B are $\overrightarrow{\omega}$-keen (resp. $\overrightarrow{\omega}_3$-keen), then $D[B]$ is also $\overrightarrow{\omega}$-keen (resp. $\overrightarrow{\omega}_3$-keen).*

Proposition 2.2 ([6] Proposition 6.3)**.** *For every tournament T, in order to determine $\overrightarrow{\omega}(T)$, it suffices to prove that there exists an optimal $\{\overrightarrow{C}_3, \overrightarrow{C}_4\}$-free $\overrightarrow{\omega}(T)$-coloring.*

A *reflexive epimorphism* from a digraph D to a digraph D' is a surjective function $\rho : V(D) \to V(D')$ such that for every $uv \in A(D)$ either $\rho(u) = \rho(v)$ or $\rho(u)\rho(v) \in A(D')$ (see [6] p. 621).

3. A special family of tournaments

In what follows, we let $\mathbf{i} = \{(i, 1), (i, 2)\}$ for $i \geq 1$. Similarly, let $\mathbf{i} + \mathbf{j} = \{(i + j, 1), (i + j, 2)\}$. Let D_{n+1} denote a digraph with vertex set $[n] \cup \{0\}$ and T_{2n+1} denote a tournament with vertex set $([n] \times \{1, 2\}) \cup \{0\}$ related as follows. Let $\pi : V(T_{2n+1}) \to V(D_{n+1})$ be a reflexive epimorphism (see [6] p. 632) from $([n] \times \{1, 2\}) \cup \{0\}$ onto $[n] \cup \{0\}$ defined by

$$\pi(v) = \begin{cases} 0 & \text{if } v = \{0\} \\ i & \text{if } v = (i, j) \end{cases} \tag{1}$$

such that the following properties hold:

(1a) $(i, 1)(i, 2) \in A(T_{2n+1})$,
(1b) if $D_{n+1}\langle\{i, j\}\rangle$ is a 2-cycle (i, j, i), then $((i, 1), (j, 1), (i, 2), (j, 2), (i, 1))$ is a 4-cycle in T_{2n+1},
(1c) if $D_{n+1}\langle\{i, j\}\rangle$ is the arc ij, then $uv \in A(T_{2n+1})$, where $u \in \mathbf{i}$ and $v \in \mathbf{j}$ (observe that u and v are ordered pairs of members of $\mathbf{i}$ and $\mathbf{j}$ respectively),
(1d) if $0i \in A(D_{n+1})$ then $0u \in A(T_{2n+1})$, where $u \in \mathbf{i}$. Similarly, if $i0 \in A(D_{n+1})$ then $u0 \in A(T_{2n+1})$.

Notice that if (1b) holds, then $T\langle\{\mathbf{i}, \mathbf{j}\}\rangle$ is the semiregular 4-tournament SR_4, and if (1c) holds then $T\langle\{\mathbf{i}, \mathbf{j}\}\rangle$ is the transitive 4-tournament TT_4 (SR_4 and TT_4 are unique up to isomorphism). The epimorphism π plays an important role in simplifying the definition of the following special family $\mathfrak{V}_n$ of tournaments.

First, we inductively define the supporting digraph D_{2m+1} as follows: D_1 is the 1-vertex tournament, D_3 is a 3-cycle with vertices in the order 0, 1, 2 and

$$D_{2m+1} = (V(D_{2m+1}), A(D_{2m+1})), \quad (m \geq 2),$$

where

$$V(D_{2m+1}) = V(D_{2m-1}) \cup \{2m - 1, 2m\},$$

$$A(D_{2m+1}) = A(D_{2m-1}) \cup \{i(2m - 1), (2m)i : i \in \mathbb{Z}_{2m-2}\} \cup \{(2m - 1)(2m), (2m - 1)(2m - 3), (2m - 2)(2m)\}$$

(see Fig. 1).