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Note

Anti-Ramsey number of matchings in hypergraphs

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ABSTRACT

A k-matching in a hypergraph is a set of k edges such that no two of these edges intersect. The *anti-Ramsey number* of a k-matching in a complete s-uniform hypergraph $\mathcal{H}$ on n vertices, denoted by $\mathrm{ar}(n, s, k)$, is the smallest integer c such that in any coloring of the edges of $\mathcal{H}$ with exactly c colors, there is a k-matching whose edges have distinct colors. The *Turán number*, denoted by $\mathrm{ex}(n, s, k)$, is the the maximum number of edges in an s-uniform hypergraph on n vertices with no k-matching. For $k \geq 3$, we conjecture that if $n > sk$, then $\mathrm{ar}(n, s, k) = \mathrm{ex}(n, s, k-1) + 2$. Also, if $n = sk$, then $\mathrm{ar}(n, s, k) = \begin{cases} \mathrm{ex}(n, s, k-1) + 2 & \text{if } k < c_s \\ \mathrm{ex}(n, s, k-1) + s + 1 & \text{if } k \geq c_s \end{cases}$, where c_s is a constant dependent on s. We prove this conjecture for $k = 2$, $k = 3$, and sufficiently large n, as well as provide upper and lower bounds.

1. Introduction

A *hypergraph* $\mathcal{H}$ consists of a set $V(\mathcal{H})$ of *vertices* and a family $\mathcal{E}(\mathcal{H})$ of nonempty subsets of $V(\mathcal{H})$ called *edges* of $\mathcal{H}$. If each edge of $\mathcal{H}$ has exactly s vertices then $\mathcal{H}$ is *s-uniform*. A *complete* s-uniform hypergraph is a hypergraph whose edge set is the set of all s-subsets of the vertex set. A *matching* is a set of edges in a (hyper)graph in which no two edges have a common vertex. We call a matching with k edges a *k-matching* and a matching containing all vertices a *perfect matching*. In an edge-coloring of a (hyper)graph $\mathcal{H}$, a sub(hyper)graph $\mathcal{F} \subseteq \mathcal{H}$ is *rainbow* if all edges of $\mathcal{F}$ have distinct colors. The *anti-Ramsey number* of a graph G, denoted by $\mathrm{ar}(G, n)$, is the minimum number of colors needed to color the edges of K_n so that, in any coloring, there exists a rainbow copy of G. The *Turán number* of a graph G, denoted by $\mathrm{ex}(n, G)$, is the maximum number of edges in a graph on n vertices that does not contain G as a subgraph. The *anti-Ramsey number* of a k-matching, denoted by $\mathrm{ar}(n, s, k)$, is the minimum number of colors needed to color the edges of a complete s-uniform hypergraph on n vertices so that there exists a rainbow k-matching in any coloring. The *Turán number* of a k-matching, denoted by $\mathrm{ex}(n, s, k)$, is the maximum number of edges in an s-uniform hypergraph on n vertices that contains no k-matching.

In 1973, Erdős, Simonovits, and Sós [6] showed that $\mathrm{ar}(K_p, n) = \mathrm{ex}(n, K_{p-1}) + 2$ for sufficiently large n. More recently, Montellano-Ballesteros and Neumann-Lara [10] extended this result to all values of n and p with $n > p \geq 3$. A history of results and open problems on this topic was given by Fujita, Magnant, and Ozeki [8]. The Turán number $\mathrm{ex}(n, 2, k)$ was determined by Erdős and Gallai [4] as

$$\mathrm{ex}(n, 2, k) = \max\left\{\binom{2k-1}{2}, \binom{k-1}{2} + (k-1)(n-k+1)\right\}$$

for $n \geq 2k$ and $k \geq 1$. Schiermeyer [11] proved that $\mathrm{ar}(n, 2, k) = \mathrm{ex}(n, 2, k-1) + 2$ for $k \geq 2$ and $n \geq 3k + 3$. Later, Chen, Li, and Tu [2] and independently Fujita, Kaneko, Schiermeyer, and Suzuki [7] showed that $\mathrm{ar}(n, 2, k) = \mathrm{ex}(n, 2, k-1) + 2$

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for $k \geq 2$ and $n \geq 2k+1$. The value

$$\mathrm{ar}(n, 2, k) = \begin{cases} \mathrm{ex}(n, 2, k-1) + 2 & \text{if } k < 7 \\ \mathrm{ex}(n, 2, k-1) + 3 & \text{if } k \geq 7 \end{cases}$$

was determined for $n = 2k$ in [2] and by Haas and the second author [9], independently.

The same ideas implying a lower bound for the anti-Ramsey number of graphs given in [6] provide a lower bound for $\mathrm{ar}(n, s, k)$.

Proposition 1. *For all n,* $\mathrm{ar}(n, s, k) \geq \mathrm{ex}(n, s, k-1) + 2$.

Proof. Let $\mathcal{H}$ be a complete s-uniform hypergraph on n vertices. Let $\mathcal{G}$ be a subhypergraph of $\mathcal{H}$ with $\mathrm{ex}(n, s, k-1)$ edges such that $\mathcal{G}$ does not contain a $(k-1)$-matching. Color each edge of $\mathcal{G}$ with distinct colors and color all of the remaining edges of $\mathcal{H}$ the same, using an additional color. If there is a rainbow k-matching in this coloring, then it uses $k-1$ edges from $\mathcal{G}$ which is a contradiction. Therefore, this coloring has no rainbow k-matching. □

For k-matchings the Turán number $\mathrm{ex}(n, s, k)$ is still not known for $k \geq 3$ and $s \geq 3$. Erdős [3] conjectured in 1965 the value of $\mathrm{ex}(n, s, k)$ as follows. Let $g(n, s, k-1)$ be the number of s-sets of $\{1, \ldots, n\}$ that intersect $\{1, \ldots, k-1\}$. By definition, $g(n, s, k-1) = \binom{n}{s} - \binom{n-k+1}{s}$.

Conjecture 2 (*Erdős [3]*)**.** *For $n \geq sk$, $s \geq 2$, and $k \geq 2$,*

$$\mathrm{ex}(n, s, k) = \max\left\{\binom{sk-1}{s}, g(n, s, k-1)\right\}. \tag{1}$$

Erdős, Ko, and Rado [5] proved that $\mathrm{ex}(n, s, 2) = \binom{n-1}{s-1} = g(n, s, 1)$ for $n \geq 2s$. This conjecture is true for $s = 2$, as shown by Erdős and Gallai [4]. Erdős [3] proved that

$$\mathrm{ex}(n, s, k) = g(n, s, k-1) = \binom{n}{s} - \binom{n-k+1}{s} \tag{2}$$

for sufficiently large n. Later, Bollobás, Daykin, and Erdős [1] sharpened this result by showing that (2) holds for $n > 2s^3(k-1)$.

In Section 2, we provide bounds on $\mathrm{ar}(n, s, k)$ and show that anti-Ramsey number and Turán number of a k-matching differ at most by a constant. In Section 3, we determine the value of $\mathrm{ar}(n, s, k)$ for $k \in \{2, 3\}$ and show that $\mathrm{ar}(n, s, k) = \mathrm{ex}(n, s, k-1) + 2$ for $k \in \{2, 3\}$ and $n > ks$. The claim also holds for $n = ks$ when $k = 3$. We conjecture that this is true for all k.

Conjecture 3. *Let $k \geq 3$. If $n > sk$, then* $\mathrm{ar}(n, s, k) = \mathrm{ex}(n, s, k-1) + 2$. *Also, if $n = sk$, then*

$$\mathrm{ar}(n, s, k) = \begin{cases} \mathrm{ex}(n, s, k-1) + 2 & \text{if } k < c_s \\ \mathrm{ex}(n, s, k-1) + s + 1 & \text{if } k \geq c_s \end{cases}$$

where c_s is a constant dependent on s.

Finally, in Section 4, we give the exact value of $\mathrm{ar}(n, s, k)$ when n is sufficiently large.

We introduce some notation for hypergraphs used in the remaining sections. For a set X, $\binom{X}{s}$ denotes all s-subsets of X. We call a hypergraph an *intersecting family* if every two edges intersect. For a vertex x in a hypergraph $\mathcal{H}$, we call the number of edges of $\mathcal{H}$ containing x the *degree* of x written $deg_{\mathcal{H}}(x)$. The maximum degree of a hypergraph $\mathcal{H}$ is denoted by $\Delta(\mathcal{H})$.

2. General bounds on the anti-Ramsey number

The following constructions provide a lower bound for $\mathrm{ar}(n, s, k)$ in Corollary 6.

Construction 4. *Let $\mathcal{H}$ be the complete s-uniform hypergraph with vertex set $\{v_1, \ldots, v_n\}$, where $n = sk$. Let $A = \{v_1, \ldots, v_{s+1}\}$ and $c = \binom{n-s-1}{s} + s$. Define a c-coloring h of $\mathcal{E}(\mathcal{H})$ as follows. For any edge $E \in \mathcal{E}$, if $v_1 \in E$, then let $h(e) = \min\{i : v_i \notin E\}$. If $E \cap A \neq \emptyset$ but $v_1 \notin E$, then let $h(E) = \min\{i : v_i \in E\}$. Assign distinct other colors to the remaining edges.*

Assume there is a rainbow perfect matching $\mathcal{M}$ in this coloring. Since $n = sk$, at least two edges of M intersect A. Let E be the edge of $\mathcal{M}$ that contains v_1. Let $j = \min\{i : v_i \notin V(E)\}$ and let E' be the edge of $\mathcal{M}$ that contains v_j. By the above construction, E and E' both have color j.