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The Path Partition Conjecture is true for some generalizations of tournaments

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ABSTRACT

The Path Partition Conjecture for digraphs states that for every digraph D, and every choice of positive integers λ_1, λ_2 such that $\lambda_1 + \lambda_2$ equals the order of a longest directed path in D, there exists a partition of D in two subdigraphs D_1, D_2 such that the order of the longest path in D_i is at most λ_i for $i = 1, 2$.

We present sufficient conditions for a digraph to satisfy the Path Partition Conjecture. Using these results, we prove that strong path mergeable, arc-locally semicomplete, strong 3-quasi-transitive, strong arc-locally in-semicomplete and strong arc-locally out-semicomplete digraphs satisfy the Path Partition Conjecture. Some previous results are generalized.

1. Introduction

Let D be a digraph, and $\lambda(D)$ denote the order of a longest directed path in D. The Gallai–Roy–Vitaver Theorem [13,16,18] states that the chromatic number of a digraph D is at most $\lambda(D)$. Laborde et al. [14] posed a conjecture that generalizes this theorem: *Every digraph has a maximal independent set that intersects every longest path*. This conjecture is a particular instance of what is called the *Path Partition Conjecture* which states the following:

Conjecture 1.1. *For every digraph D and any choice of positive integers* λ_1 *and* λ_2 *with* $\lambda(D) = \lambda_1 + \lambda_2$*, there exists a partition of D into two subdigraphs* D_1 *and* D_2 *such that* $\lambda(D_i) \leq \lambda_i$ *for* $i = 1, 2$.

Similar partition problems have been solved for undirected graphs in [15,17] considering parameters such as the maximum degree Δ or the minimum degree δ. In [6] this conjecture is proved for several generalizations of tournaments, namely quasi-transitive, extended semicomplete and locally in-semicomplete digraphs. Other extensions of the Laborde, Payan and Xuong conjecture (such as *find a maximal independent set that intersects every non-augmentable path*) were proved for line digraphs, arc-locally semicomplete, quasi-antiarc-transitive, quasi-transitive, path-mergeable, locally in-semicomplete, locally out-semicomplete, semicomplete k-partite digraphs [12], and more recently for arc-locally in-semicomplete, arc-locally out-semicomplete and quasi-arc-transitive digraphs [20]. In the present paper, we prove that strong path mergeable, arc-locally semicomplete, strong 3-quasi-transitive, strong arc-locally in-semicomplete and arc-locally out-semicomplete digraphs, satisfy the Path Partition Conjecture.

Theorem 5.1, our main result in Section 5, allow us to use digraphs which satisfy the conditions of the Path Partition Conjecture to construct other digraphs that also satisfy the conditions of this seemingly difficult conjecture. We take

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advantage of this useful result combined with recent characterizations of certain classes of digraphs (Sections 6–8) to prove that digraphs belonging to these classes satisfy Conjecture 1.1.

Path mergeable digraphs extend several generalizations of tournaments such as locally in-semicomplete, locally out-semicomplete, locally semicomplete digraphs (classes for which the Path Partition has been proved); also there are infinite families of path mergeable digraphs that are not contained in any of the previously mentioned classes. Arc-locally semicomplete, strong arc-locally in-semicomplete, strong arc-locally out-semicomplete and strong 3-quasi-transitive digraphs were introduced in [2,19].

2. Terminology and notation

We only consider finite digraphs without loops or multiple arcs. For general terminology, consult [4,7]. Let $D = (V(D), A(D))$ be a *digraph*, the *order* of D (denoted by $|D|$), is the cardinality of $V(D)$. Consider two vertices $u, v \in V(D)$, we will denote $u \to v$, $\overrightarrow{uv}$ or simply uv when there exists an arc from u to v. We say that x and y are *adjacent* if $\overrightarrow{xy}$ or $\overrightarrow{yx}$. Consider $S, T \subset V(D)$, $D[S]$ is the subdigraph induced by S (if H is graph or a digraph having $V(H) \subset V(D)$, then we will write $D[H]$ instead of $D[V(H)]$). An ST-arc is an arc $\overrightarrow{st}$ with $s \in S$ and $t \in T$. We say that S *dominates* T (denoted by $S \mapsto T$) if $st \in A(D)$ for every $s \in S$ and every $t \in T$.

In the context of digraphs, we always consider a *path* (*cycle*) as a directed path (directed cycle). An xy-path is a path whose initial vertex is x and its final vertex is y. An ST-path is an st-path such that $s \in S$ and $t \in T$. The *order* (*length*) of a path is the number of vertices (arcs) belonging to this path. Sometimes we use $\lambda(S)$ instead of writing $\lambda(D[S])$. If $P = (x_1, \dots, x_n)$ and $Q = (y_1, \dots, y_m)$ are paths in D, we denote the subpath $(x_i, \dots, x_j)$ by x_iPx_j and if $x_n \to y_1$ the *concatenation* between P and Q will be denoted simply by PQ.

The *underlying graph* (denoted by $UG(D)$) of a digraph D is the undirected graph obtained by replacing arcs or directed cycles of order 2, by edges. The *strong component* digraph of D is the acyclic digraph $SC(D)$ where $V(SC(D)) = \{S_1, S_2, \dots, S_r\}$ is the set whose elements are the strong components of D and $A(SC(D)) = \{S_iS_j | \overrightarrow{s_is_j} \in A(D)$, for some $s_i \in S_i$ and $s_j \in S_j\}$. An (A, B) *partition* of $V(D)$ is a bipartition of $V(D)$, in which $A \cup B = V(D)$ and $A \cap B = \emptyset$. For a positive integer n, let E_n be the digraph with n vertices and no arcs. E_n is called the independent set of order n.

3. Digraphs with Hamiltonian blocks

A *cut vertex* of a graph is one whose removal increases the number of components. A *nonseparable* graph is connected, nontrivial and has no cut vertices. A block of a graph is a maximal nonseparable subgraph. If P is a path in a digraph D and B is any block in $UG(D)$, then the intersection of P with $D[B]$ is either empty or a path. Considering the fact that every graph can be thought as a symmetrically directed graph, the next result is a generalization of a theorem due to Broere et al. [8].

Theorem 3.1. *Suppose D is a digraph such that $D[X]$ is a Hamiltonian for every block X of the underlying graph $UG(D)$. Then there exists a partition (A, B) of $V(D)$ such that $\lambda(D[A]) \le \lambda_1$ and $\lambda(D[B]) \le \lambda_2$, for any two positive integers $\lambda_1 \ge \lambda_2$ such that $\lambda_1 + \lambda_2 = \lambda(D)$.*

Proof. Suppose λ_1, λ_2 are positive integers such that $\lambda(D) = \lambda_1 + \lambda_2$ and $\lambda_1 \ge \lambda_2$. We may assume that D is connected. Since $D[X]$ is Hamiltonian for every block X of $UG(D)$, there is no block B in the underlying graph $UG(D)$ such that $D[B]$ is a directed path of order 2. By this observation, there is a path in D of order at least $|X| + |Y|$, where X and Y are any two blocks in $UG(D)$. We consider two cases.

Case 1. *Some block X in $UG(D)$ has more than λ_1 vertices.*

Let Y be any block of $UG(D)$ other than X. Then there is a path in D that contains $V(X) \cup V(Y)$. (Notice that if $Y = \emptyset$ it is trivial.) If $|Y| \ge \lambda_2 + 1$ then, since X and Y have at most one common vertex, $|P| > \lambda_1 + \lambda_2$, contradicting our assumption on $\lambda(D)$. Hence every block other than X has at most λ_2 vertices. Now put λ_1 vertices of X into A and the remaining vertices of X into B. (Notice that $|X| \le \lambda = \lambda_1 + \lambda_2$ as $D[X]$ is Hamiltonian.) Then repeat the following procedure until all vertices of D have been distributed between A and B.

Let Z be any block that has not yet been partitioned but that shares a vertex v with a block whose vertices have already been distributed among A and B (since the block–cut-vertex structure of a graph is acyclic, any block that has not yet been partitioned will share at most one vertex with that part of the graph whose vertices have already been partitioned). If v is in A (resp. v is in B), put all the vertices of $Z - v$ in B (resp. in A).

It is clear from the procedure that every path in $D[A]$ and every path in $D[B]$ is contained in a single block of $UG(D)$. Since all the blocks except X have order at most λ_2, it follows that $\lambda(D[A]) \le \lambda_1$ and $\lambda(D[B]) \le \lambda_2$.

Case 2. *Every block of $UG(D)$ has at most λ_1 vertices.*

Start with any block X of $UG(D)$ and put $\min\{\lambda_2, |X|\}$ of its vertices into B and the remaining vertices into A. Then repeat the following procedure until all vertices of D have been distributed between A and B.

Let Y be any block that has not yet been partitioned but that shares a vertex v with a block whose vertices have already been distributed among A and B. If v is in B, then put all the vertices of $Y - v$ into A. If v is in A, then put $\min\{|Y - v|, \lambda_2\}$ vertices of Y into B and the remainder into A (see Fig. 1).