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Precise location of vertices on Hamiltonian cycles

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ABSTRACT

Given $k \geq 2$ fixed positive integers $p_1, p_2, \ldots, p_{k-1} \geq 2$, and k vertices $\{x_1, x_2, \ldots, x_k\}$, let G be a simple graph of sufficiently large order n. It is proved that if $\delta(G) \geq (n + 2k - 2)/2$, then there is a Hamiltonian cycle C of G containing the vertices in order such that the distance along C is $d_C(x_i, x_{i+1}) = p_i$ for $1 \leq i \leq k - 1$. Also, let $\{(x_i, y_i)|1 \leq i \leq k\}$ be a set of k disjoint pairs of vertices and a graph of sufficiently large graph n and $p_1, p_2, \ldots, p_k \geq 2$ for $k \geq 2$ fixed positive integers. It will be proved that if $\delta(G) \geq (n + 3k - 1)/2$, then there are k vertex disjoint paths $P_i(x_i, y_i)$ of length p_i for $1 \leq i \leq k$.

1. Introduction

Given an ordered set of vertices $S = \{x_1, x_2, \ldots, x_k\}$ in a graph, there are a series of results giving minimum degree conditions that imply the existence of a Hamiltonian cycle such that the vertices in S are located in order on the cycle with restrictions on the distance between consecutive vertices of S. Examples include results by Kaneko and Yoshimoto [6], Sárközy and Selkow [9], and Faudree, Gould, Jacobson, and Magnant [3]. In each of these results the distances on the Hamiltonian cycle was close, by not precise relative to the predetermined objective. In the case of pairs of vertices, there are results in which the distance between the vertices is precise by Faudree and Li [5] and Faudree, Lehel, and Yoshimoto [4]. The objective is to replace 2 by a fixed number $k \geq 3$ of vertices that can be placed on a Hamiltonian cycle at precise predetermined distances. However, in this case the distances will not be a positive fraction of the order of the Hamiltonian cycle, as was true in some of the previous cases.

We deal only with finite simple graphs and our notation generally follows the notation of Chartrand and Lesniak in [1]. The connectivity of a graph G will be denoted by $\kappa(G)$ and the independence number by $\alpha(G)$. A path (or cycle) with an ordered set of vertices $\{x_1, x_2, \ldots, x_k\}$ will be denoted by $(x_1, x_2, \ldots, x_k)$ (or $(x_1, x_2, \ldots, x_k, x_1)$). If x_i is a vertex of a cycle (path) then x_i^+ will denote the successor x_{i+1}, and if S is a set of its vertices, then S^+ will denote the set of all successors of the vertices of S. The set S^- of predecessors is defined similarly. Given a cycle C containing vertices x and y, let $d_C(x, y)$ denote the distance between x and y on the cycle C. The set of all adjacencies of a vertex $v \in G$ in $S \subset G$ will be denoted by $N_S(v)$, and we set $d_S(v) = |N_S(v)|$.

A graph G of order n is *panconnected* if between each pair of vertices x and y of G there is a path $P_i(x, y)$ of length i for each $d_G(x, y) \leq i < n$. Williamson [10] gave a minimum degree condition that implies a graph is panconnected.

Theorem 1 ([10]). *If G is a graph of order n with $\delta(G) \geq n/2 + 1$, then for any $2 \leq k \leq n - 1$ and for any vertices x and y, G has a path from x to y of length k.* □

We prove the following, which in some sense generalizes the result of Williamson [10].

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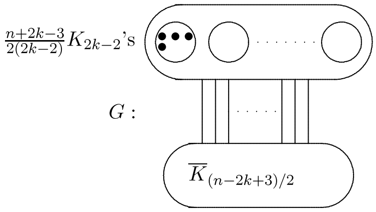

$$G = \overline{K}_{(n-2k+3)/2} + ((n+2k-3)/2(2k-2))(K_{2k-2})$$
$$\delta(G) = (n+2k-3)/2$$

Fig. 1. Panconnected example.

Theorem 2. *Let $\{p_1, p_2, \ldots, p_{k-1}\}$ be a set of $k-1$ integers each at least 2, and $\{x_1, x_2, \ldots, x_k\}$ an ordered set of k vertices of a graph G of order n. If*

$$\delta(G) \geq (n+2k-2)/2,$$

then, there is an $n_0 = n_0(k, p_1, p_2, \ldots, p_{k-1})$ such that if $n \geq n_0$ there is a path P such that $d_P(x_i, x_{i+1}) = p_i$ for $1 \leq i \leq k-1$.

The graph G in Fig. 1 implies that the degree condition in Theorem 2 is sharp, since if k vertices are all in one of the complete graphs K_{2k-2} of G, there does not exist $k-1$ paths of length 3 between consecutive pairs of these vertices.

The following conjecture of Enomoto [8] created research interest in placing vertices at precise distances on Hamiltonian cycles.

Conjecture 1 (*[8]*). *If G is a graph of order $n \geq 3$ and $\delta(G) \geq n/2 + 1$, then for any pair of vertices x, y in G, there is a Hamiltonian cycle C of G such that $d_C(x, y) = \lfloor n/2 \rfloor$.*

The following result of Faudree, Lehel, and Yoshimoto [4], which was motivated by the conjecture of Enomoto [8], deals with locating a pair of vertices at a precise distance on a Hamiltonian cycle.

Theorem 3 (*[4]*). *Let $k \geq 2$ be a fixed positive integer. If G is a graph of order $n \geq 6k$ and $\delta(G) \geq (n+2)/2$, then for any vertices x and y, G has a Hamiltonian cycle C such that $d_C(x, y) = k$.*

This result along with Theorem 2 will be generalized in some sense in the following theorem, which will be proved.

Theorem 4. *Let $\{p_1, p_2, \ldots, p_{k-1}\}$ be a set of $k-1$ integers and $\{x_1, x_2, \ldots, x_k\}$ a fixed set of k ordered vertices in a graph G of order n. If*

$$\delta(G) \geq (n+2k-2)/2,$$

then, there is a $n_0 = n_0(k, p_1, p_2, \ldots, p_{k-1})$ such that if $n \geq n_0$, there is a Hamiltonian cycle C such that $d_C(x_i, x_{i+1}) = p_i$ for $1 \leq i \leq k-1$.

Just as in the case of Theorem 2, Fig. 1 verifies that the degree condition of Theorem 4 is sharp.

For $k \geq 1$ a graph G is *k-linked*, if given any set of k disjoint pairs of vertices $\{(x_i, y_i) \mid 1 \leq i \leq k\}$, there exist k vertex disjoint paths $P_1(x_1, y_1), P_2(x_2, y_2), \ldots, P_k(x_k, y_k)$ between the k pairs of vertices.

The next theorem to be proved is a natural companion of Theorem 2.

Theorem 5. *Let $\{(x_i, y_i) \mid 1 \leq i \leq k\}$ be a set of k disjoint pairs of vertices in a graph G of order n, and let $p_1, p_2, \ldots, p_k \geq 2$ and $k \geq 2$ be fixed positive integers. If*

$$\delta(G) \geq (n+3k-1)/2,$$

then, there is an $n_0 = n_0(k, p_1, p_2, \ldots, p_k)$ such that if $n \geq n_0$, there are k vertex disjoint paths $P_i(x_i, y_i)$ of length p_i for $1 \leq i \leq k$.

The graph G in Fig. 2 that follows implies that the degree condition in Theorem 5 is sharp, since if the $2k$ vertices in the linkage are all in one of the K_{3k-1} complete graphs of G, there does not exist k paths of length 3 between the k pairs of these vertices.

2. Proofs

Proof of Theorem 2. Let $\{x_1, x_2, \ldots, x_k\}$ denote the k vertices and $p_1, p_2, \ldots, p_{k-1}$ the $k-1$ integers each at least 2 with $p = (\sum_{i=1}^{k-1} p_i) + 1$. The proof will be by double induction, first on k and then on p. In the case when $k = 2$, the result is just Theorem 1. The smallest value of p is $2(k-1)+1$. Observe that each x_i has at least $(n+2k-2)/2 - (k-1) = n/2$