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Describing $(d-2)$-stars at d-vertices, $d \leq 5$, in normal plane maps

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ABSTRACT

We prove that every normal plane map has a $(3, 10^-)$-edge, or a $(5^-, 4, 9^-)$-path, or a $(6, 4, 8^-)$-path, or a $(7, 4, 7)$-path, or a $(5; 4, 5, 5)$-star, or a $(5; 5, b, c)$-star with $5 \leq b \leq 6$ and $5 \leq c \leq 7$, or a $(5; 6, 6, 6)$-star. Moreover, none of the above options can be strengthened or dropped.

In particular, this extends or strengthens several known results and disproves a related conjecture of Harant and Jendrol' (2007) [10].

1. Introduction

We shall use the following terminology. The degree of a vertex v or a face f, that is the number of edges incident with v or f (loops and cut-edges are counted twice), is denoted by $d(v)$ or $r(f)$, respectively. By $v_1, \ldots, v_{d(v)}$ we denote the neighbors of v in a cyclic order round v. A *k-vertex* is a vertex v with $d(v) = k$. By k^+ or k^- we denote any integer not smaller or not greater than k, respectively. Hence, a k^+-vertex v has $d(v) \geq k$, etc. A normal plane map (NPM) is a plane pseudograph in which loops and multiple edges are allowed, but the degree of each vertex and face is at least 3. Let δ be the minimum vertex degree.

By M_5 denote an NPM with $\delta(M_5) = 5$. Back in 1904, Wernicke [18] proved that every M_5 contains a 5-vertex adjacent to a 6^--vertex, and Franklin [8] strengthened this to the existence of at least two 6^--neighbors. In 1940, Lebesgue [16] showed that every M_5 has a 5-vertex adjacent to three 8^--vertices.

By $w(S_k)$, we denote *the minimum weight* (i.e., the minimum degree-sum) of a k-star centered at a 5-vertex in M_5. The following precise bounds hold: $w(S_1) \leq 11$ [18], $w(S_2) \leq 17$ [8], $w(S_3) \leq 23$ [13], and $w(S_4) \leq 30$ (Borodin and Woodall [7], strengthening the bound $w(S_4) \leq 39$ given in [13]). Note that $w(S_3) \leq 23$ readily implies that $w(S_2) \leq 17$ and easily follows from $w(S_4) \leq 30$.

An *(a, b)-edge* (*(a, b^-)-edge*) is an edge xy such that $d(x) = a$ and $d(y) = b$ ($d(x) = a$ and $d(y) \leq b$). An *$(a, 4, b)$-path* (*$(a^-, 4, b)$-path*, etc.) is a path xyz such that $d(x) = a$, $d(y) = 4$, and $d(z) = b$ (or $d(x) \leq a$, $d(y) = 4$, and $d(z) = b$, respectively). A *$(5; a, b, c)$-star* is a 3-star centered at a 5-vertex v such that v is adjacent to an a-vertex, a b-vertex, and a c-vertex.

It follows from Lebesgue's results given in [16] that each normal plane map has an edge $e = uw$ of weight $w(e) = d(u)+d(w)$ at most 14 (more specifically, a $(3, 11^-)$-, or $(4, 7^-)$-, or $(5, 6^-)$-edge). For 3-connected plane graphs, Kotzig [15] (in 1955) proved a precise result: there is an e with $w(e) \leq 13$.

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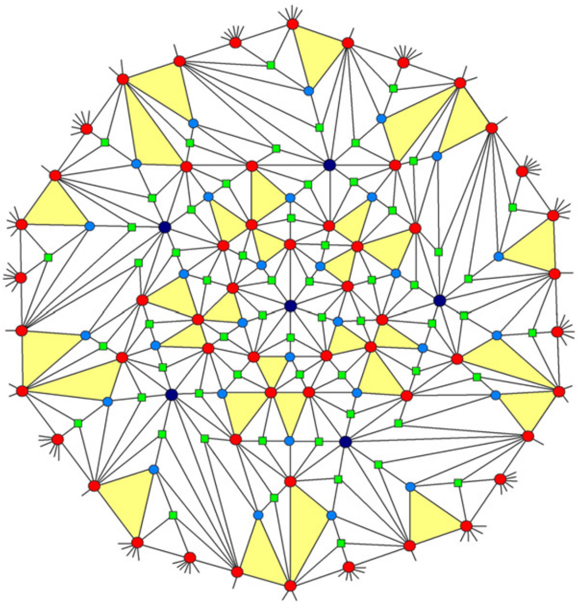

Fig. 1. A counterexample to Conjecture 1.

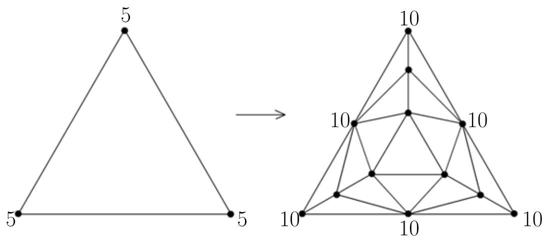

Fig. 2. A different counterexample to Conjecture 1.

In 1972, Erdős (see [9]) conjectured that Kotzig's bound 13 holds for all planar graphs with $\delta \geq 3$. Barnette (see [9]) announced having proved this conjecture, but the proof has never appeared in print. The first published proof of the conjecture of Erdős belongs to Borodin [2]. More generally, Borodin proved [3–6] that every NPM contains a $(3, 10^-)$-, or $(4, 7^-)$-, or $(5, 6^-)$-edge (as easy corollaries of some stronger structural facts that had applications to coloring). The same fact was later on deduced by Jendrol' [11,12] from some stronger structural results.

Van den Heuvel and McGuinness [17] proved that every planar graph with $\delta \geq 3$ contains a $(3, 11^-)$-edge, or a $(7^-, 4, 11^-)$-path, or a $(5; 6^-, 7^-, 11^-)$-star.

Balogh et al. [1] proved that every simple planar graph contains a vertex of degree at most 5 which is adjacent to at most two 11^+-vertices.

Harant and Jendrol' [10,14, Theorem 3.8] absorbed most of the above mentioned results by proving that every planar graph G with $\delta(G) \geq 3$ contains one of the following configurations: (i) a $(3, 10^-)$-edge, or (ii) an $(a, 4, b)$-path, where $a = 4$ and $4 \leq b \leq 10$, or $a = 5$ and $5 \leq b \leq 9$, or $6 \leq a \leq 7$ and $6 \leq b \leq 8$, or (iii) a $(5; a, b, c)$-star, where $4 \leq a \leq 5$, $5 \leq b \leq 6$, and $5 \leq c \leq 7$, or $a = b = c = 6$.

Conjecture 1 (*Harant and Jendrol' [10]*)**.** *Every simple planar graph G with $\delta(G) \geq 3$ contains*

(i) *a $(3, 10^-)$-edge, or*
(ii) *an $(a, 4, b)$-path, where $a = 4$ and $4 \leq b \leq 9$, or $a = 5$ and $5 \leq b \leq 8$, or $a = 6$ and $6 \leq b \leq 8$, or $a = b = 7$, or*
(iii) *a $(5; a, b, c)$-star, where $a = 5$, $5 \leq b \leq 6$, and $5 \leq c \leq 7$, or $a = b = c = 6$.*

However, Conjecture 1 turns out to be wrong, and we now present two kinds of counterexamples to it. In Fig. 1, we see a half of a plane triangulation with the following properties: each vertex has degree 4, 5, 9, or 10, no two 4-vertices are adjacent, no 4-vertex is adjacent to two 5-vertices, and no 5-vertex is adjacent to a 5-vertex. Fig. 2 shows how to transform the icosahedron into a triangulation with all vertices having degree 4, 5, or 10 and such that no 4-vertex is adjacent to two 5-vertices.

The main purpose of this paper is to precisely describe $(d-2)$-stars at d-vertices, $d \leq 5$, in normal plane maps (in particular, in planar graphs G with $\delta(G) \geq 3$) as follows: