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Note

Describing 4-stars at 5-vertices in normal plane maps with minimum degree 5

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ABSTRACT

Lebesgue (1940) [13] proved that each plane normal map M_5 with minimum degree 5 has a 5-vertex such that the degree-sum (the weight) of its every four neighbors is at most 26. In other words, every M_5 has a 4-star of weight at most 31 centered at a 5-vertex. Borodin–Woodall (1998) [3] improved this 31 to the tight bound 30.

We refine the tightness of Borodin–Woodall's bound 30 by presenting six M_5s such that (1) every 4-star at a 5-vertex in them has weight at least 30 and (2) for each of the six possible types (5, 5, 5, 10), (5, 5, 6, 9), (5, 5, 7, 8), (5, 6, 6, 8), (5, 6, 7, 7), and (6, 6, 6, 7) of 4-stars with weight 30, the 4-stars of this type at 5-vertices appear in precisely one of these six M_5s.

1. Introduction

The degree of a vertex v or a face f is the number of edges incident with v or f, where loops and cut-edges are counted twice, respectively. A normal plane map (NPM) is a plane pseudograph in which loops and multiple edges are allowed, but the degree of each vertex and face is at least three.

The degree of a vertex v is denoted by $d(v)$. A vertex v is a *k-vertex* if $d(v) = k$. A k^+*-vertex* (k^-*-vertex*) is one with degree at least k (at most k). An NPM with minimum degree 5 is denoted by M_5. The *weight* $w(H)$ of a subgraph H of a map M is the degree-sum of the vertices of H in M. A k-star $S_k(v)$ with the central vertex v is *minor* if $d(v) \le 5$. All stars considered in this note are minor. A minor star $S_k(v)$ with rays $v_1, \ldots, v_k$ is a *k-star of type* $(p_1, \ldots, p_k)$ or a $(p_1, \ldots, p_k)$*-star* if $\{d(v_1), \ldots, d(v_k)\}$ is majorized by the vector $(p_1, \ldots, p_k)$ with $(p_1 \le \ldots P_k)$. By $w(S_k)$ we denote the minimum integer W such that the weight of every minor k-star in a given NPM is at most W.

In 1904, Wernicke [14] proved that every M_5 has a 5-vertex adjacent to a 6^--vertex. This result was strengthened by Franklin [8] in 1922 to the existence of a 5-vertex with two 6^--neighbors. In 1940, Lebesgue [13, p. 36] gave an approximate description of the neighborhoods of 5-vertices in M_5s. In particular, this description implies the results in [14,8] and shows that there is a 5-vertex with three 8^--neighbors. From Lebesgue [13, p. 36] we can easily deduce the following rough description of minor 5-stars in M_5.

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Theorem 1 (*Lebesgue [13]*)**.** *Every normal plane map with minimum degree 5 has a minor 5-star of one of the following types*[1]*:*

$$
\begin{array}{lll}
(5,5,5,7,41), & (5,5,6,7,27)^*, & (5,6,6,7,11),\\
(5,5,5,8,23), & (5,5,6,8,15)^*, & (5,6,6,8,10),\\
(5,5,5,9,17), & (5,5,6,9,11), & (5,6,7,7,8),\\
(5,5,5,10,14), & (5,5,7,7,13), & (6,6,6,6,11),\\
(5,5,5,11,13), & (5,5,7,8,10), & (6,6,6,7,9),\\
(5,5,6,6,\infty), & (5,6,6,6,17), & (6,6,7,7,7).
\end{array}
$$

The bounds $w(S_1) \le 11$ (Wernicke [14]) and $w(S_2) \le 17$ (Franklin [8]) for all M_5s are tight. It follows from Theorem 1 that $w(S_3) \le 24$ and $w(S_4) \le 31$ for every M_5, which was improved much later to the following tight bounds: $w(S_3) \le 23$ (Jendrol'–Madaras [11]) and $w(S_4) \le 30$ (Borodin–Woodall [3]). Note that $w(S_3) \le 23$ easily implies $w(S_2) \le 17$ and immediately follows from $w(S_4) \le 30$ (it suffices to delete a vertex of maximum degree from a minor star of the minimum weight).

Jendrol' and Madaras [11] completely described minor 3-stars in M_5s as follows.

Theorem 2 (*Jendrol'–Madaras [11]*)**.** *Every normal plane map with minimum degree 5 has a minor 3-star of one of the types* $(6,6,6)$ *and* $(5,6,7)$*, where each parameter is tight.*

The purpose of our note is to give a similar description for minor 4-stars that implies Theorem 2 and shows that the tight bound $w(S_4) \le 30$ in Borodin–Woodall [3] is attained by any (p_1, p_2, p_3, p_4)-star with $p_1 + p_2 + p_3 + p_4 = 30$.

Theorem 3. *Every normal plane map with minimum degree 5 has a minor 4-star of one of the following types:*
(Ta) $(6,6,6,7)$;
(Tb) $(5,6,7,7)$;
(Tc) $(5,6,6,8)$;
(Td) $(5,5,7,8)$;
(Te) $(5,5,6,9)$;
(Tf) $(5,5,5,10)$.
Moreover, each parameter in (Ta)–(Tf) *is tight, as shown by certain plane triangulations without loops and multiple edges.*

In other words, we refine the tightness of Borodin–Woodall's bound 30 by presenting six M_5s such that (1) every minor 4-star in them has weight at least 30 and (2) all minor 4-stars of each of the six possible types of 4-stars with $w = 30$ appear in precisely one of these six M_5s.

The following problem arises naturally from Lebesgue's Theorem 1 and subsequent results in [3,11].

Problem 1. Find a complete description of 5-stars centered at 5-vertices in normal plane maps with minimum degree 5.

So Problem 1 asks for a best possible version of Lebesgue's Theorem 1 (it is not excluded that there are more than one such versions, but each of them should imply Theorem 3). Differently put, Problem 1 consists in decreasing the vector

$$\Lambda_{18} = (41, 23, 17, 14, 13, \infty, 27, 15, 11, 13, 10, 17, 11, 10, 8, 11, 9, 7)$$

of the fifth components of the 18 terms in Theorem 1 to a minimal vector (or vectors).

In fact, only one term, $(5,5,6,6,\infty)$, in Theorem 1 is known to us to be tight. Take three concentric n-cycles $C^i = v_1^i \dots v_N^i$, where N is large and $1 \le i \le 3$, and join C^2 with C^1 by edges $v_j^2 v_j^1$ and $v_j^2 v_{j+1}^1$ whenever $1 \le j \le N$ (addition modulo N). The same is done with C^2 and C^3. Finally, join all vertices of C^1 to a new N-vertex and do the same with C^3. As a result, every 5-vertex is adjacent to an N-vertex, two 5-vertices, and two 6-vertices.

Note that Jendrol' and Madaras [11] suggested a similar construction in which every 5-vertex is adjacent to an N-vertex and four 5-vertices and which shows that $w(S_5)$ is unbounded in M_5s.

On the other hand, it follows from Theorem 1 that if an M_5 has no $(5,5,6,6)$-stars, then $w(S_5) \le 68$. Recently, Borodin, Ivanova, and Jensen [7] lowered this bound of 68 to 55, but further progress in this direction is not excluded (the lower bound in [7] is 48).

For arbitrary NPMs, the following results concerning $(d-2)$-stars at d-vertices, $d \le 5$, are known. Van den Heuvel and McGuinness [10] proved (in particular) that there is an $S_k(v)$ such that either $w(S_1(v)) \le 14$ with $d(v) = 3$, or $w(S_2(v)) \le 22$ with $d(v) = 4$, or $w(S_3(v)) \le 29$ with $d(v) = 5$. Balogh et al. [1] proved that there is a 5^--vertex adjacent to at most two 11^+-vertices. Harant and Jendrol' [9] strengthened these results by proving (in particular) that we always have an $S_k(v)$ such that either $w(S_1(v)) \le 13$ with $d(v) = 3$, or $w(S_2(v)) \le 19$ with $d(v) = 4$, or $w(S_3(v)) \le 23$ with $d(v) = 5$. Recently, we obtained [6] an exhaustive description of minor $(d-2)$-stars in all NPMs, which refines the description given in Harant–Jendrol' [9].

For the more difficult problem of describing $(d-1)$-stars at d-vertices, $d \le 5$, in all NPMs, only approximate or partial results have been achieved as yet; see Borodin et al. [4,5] and a recent survey by Jendrol' and Voss [12].

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1 There are misprints in Lebesgue [13, p. 36] concerning the two terms labeled by asterisk.