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Intersecting longest paths[☆]

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ABSTRACT

In 1966, Gallai asked whether every connected graph has a vertex that is common to all longest paths. The answer to this question is negative. We prove that the answer is positive for outerplanar graphs and 2-trees. Another related question was raised by Zamfirescu in the 1980s: *Do any three longest paths in a connected graph have a vertex in common?* The answer to this question is unknown. We prove that for connected graphs in which all nontrivial blocks are Hamiltonian the answer is affirmative. Finally, we state a conjecture and explain how it relates to the three longest paths question.
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1. Introduction

In 1966, at the Tihany colloquium on graph theory, Gallai [5] asked whether every connected graph has a vertex that appears in all longest paths. A few years later, Walther [16] answered this question negatively by exhibiting a counterexample on 25 vertices. Later, Walther [18] and Zamfirescu [21] independently found a smaller counterexample on 12 vertices. See Fig. 1. Note also that hypotraceable graphs constitute a large class of counterexamples. (A graph G is hypotraceable if G has no Hamiltonian path but every vertex-deleted subgraph $G - v$ has.) Thomassen [12] proved the existence of infinitely many planar hypotraceable graphs.

On the other hand, there are classes of graphs for which the answer to Gallai's question is positive. For example, in a tree, all longest paths must contain its center(s). In 1990, Klavžar and Petkovšek [9] proved that the answer is also positive for split graphs, cacti, and some other classes of graphs. More recently, Balister, Győri, Lehel, and Schelp [2] established a similar result for the class of circular arc graphs.

Since nonempty intersection of all longest paths seems to be a property exhibited by few classes of graphs, it is natural to consider the intersection of a smaller number of longest paths. While it is easy to prove that every two longest paths share a vertex, it is not known whether every three longest paths also share a vertex. This question has been asked by Zamfirescu [14,22] since the 1980s [23]. It was mentioned at the 15th British Combinatorial Conference [3], and it appears as a conjecture in [7], and as an open problem in the list collected and maintained by West [19].

Conjecture 1. *For every connected graph, any three of its longest paths have a common vertex.*

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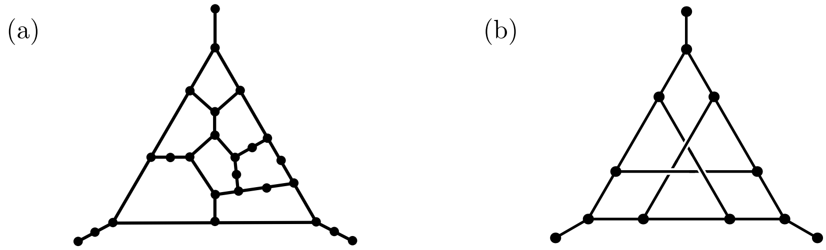

Fig. 1. (a) The graph of Walther on 25 vertices. (b) The smaller graph of Walther and Zamfirescu.

Excluding results regarding the intersection of all longest paths, the only known progress on Conjecture 1 was obtained by Axenovich, who proved the following result.

Theorem 1 (*Axenovich, Theorem 1 in [1]*). *If G is a connected outerplanar graph, then any three longest paths in G have a common vertex.*

In this paper, we prove three results. First, in Section 3, we show that if G is a connected outerplanar graph, then G has a vertex common to all longest paths. Then, in Section 4, we prove a similar result for 2-trees. In Section 5, we prove that if all nontrivial blocks of a connected graph are Hamiltonian, then Conjecture 1 holds. Notice that the first and the last results generalize Theorem 1, each in a different way. Finally, as an attempt to attack Conjecture 1, we propose a weaker conjecture on three paths with fixed extremes, presented in Section 6.

A preliminary version of the present paper, containing only part of the results to be shown here, appeared in the EuroComb 2011 proceedings [4].

2. Preliminaries

In this paper, all graphs are undirected and simple. A graph is *outerplanar* if it has an embedding in the plane with the property that all vertices belong to the boundary of its outer face, the unbounded face.

A graph G is k-*outerplanar* for $k = 1$ if G is outerplanar, and for $k \geq 2$ if there is an embedding of G in the plane such that the removal of all vertices belonging to (the boundary of) the outer face results in a $(k-1)$-outerplanar graph. For instance, the graph in Fig. 1(a) is 2-outerplanar.

We denote paths by capital letters (e.g., P, Q, R), and denote their lengths by the corresponding lowercase letters (p, q, r). To distinguish the direction in which a path P is traversed, we call one of its endpoints the *origin* of P. We write P^{-1} to indicate the reverse of P. A uv-*path* is a path between u and v whose origin is u. If P is a uv-path and R is a path with origin v, then $P \cdot R$ denotes the concatenation of P and R. If x and y are vertices of P, we denote by P_{xy} the section of P from x to y. Analogously, if C is a cycle, then c denotes its length; and if C is embedded in the plane and x and y are vertices of C, we denote by C_{xy} the section of C that goes clockwise from x to y.

For a nontrivial block B of a graph G, we say that a nontrivial path in G is a *pendent path* of B if it intersects B precisely in its origin and it is maximal. When P is a path and B is a block clear from the context, we let $\widetilde{P}$ denote the restriction of P to B, that is, $\widetilde{P} = B \cap P$.

Klavžar and Petkovšek [9] characterized graphs that have a vertex common to all longest paths in terms of a condition that is local to blocks.

Theorem 2 (*Klavžar and Petkovšek, Theorem 3.3 in [9]*). *Let G be a connected graph and $\mathscr{P}$ be the set of all longest paths in G. There exists a vertex common to all paths in $\mathscr{P}$ if and only if, for every block B of G, all longest paths in $\mathscr{P}$ with at least one edge in B have a common vertex.*

In fact, the proof (of the sufficiency of the condition on the blocks of G) presented by Klavžar and Petkovšek actually demonstrates the following theorem, which is stronger than the result they originally stated.

Theorem 3. *Let G be a connected graph and $\mathscr{P}$ be a set of longest paths. If there is no vertex common to all paths in $\mathscr{P}$, then there exists a block B of G containing at least one edge from each path in $\mathscr{P}$.*

Note that the set $\mathscr{P}$ in the previous theorem is not necessarily the set of all longest paths of the graph, as in the statement of Theorem 2.

In view of this result, to show that all longest paths share a vertex in a graph G, we need only prove that, for every block B of G, all longest paths using at least one edge from B must have a common vertex.

Notice Theorem 2 implies that, if G is a cactus, then all longest paths in G have a common vertex.

3. Longest paths in outerplanar graphs

A larger class of graphs containing the cacti is the class of outerplanar graphs. As we mentioned already, Klavžar and Petkovšek [9] proved that the answer to Gallai's question is positive for cacti, while Axenovich [1] proved that Conjecture 1

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