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Lifting automorphisms along abelian regular coverings of graphs

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ABSTRACT

This article proposes an effective criterion for lifting automorphisms along regular coverings of graphs, with the covering transformation group being any finite abelian group.

1. Introduction

First let us recall some graph theory terminology. For more one can refer to [1–3,6,8].

We assume that all graphs are finite, connected, simple (that is, having no loops or parallel edges) and undirected. More precisely, a graph is viewed as a finite one-dimensional simplicial complex (see [4, p. 104]) with orientation forgotten. For a graph Γ, denote an edge connecting the vertices u, v by $\{u, v\}$; each edge $\{u, v\}$ gives rise to a pair of opposite arcs (u, v) and (v, u). Use $V(\Gamma), E(\Gamma), A(\Gamma)$ to denote the set of vertices, edges, arcs of Γ, respectively, and use $\mathrm{Aut}(\Gamma)$ for the automorphism group.

A *covering* of a graph $\pi : \tilde{\Gamma} \to \Gamma$ is a simplicial covering map of one-dimensional simplicial complexes. It is called *regular* (or an *A-covering*) if there is a subgroup A of the covering transformation group K that acts regularly on each fiber. Moreover, $A = K$ if Γ is connected.

Let A be a finite group. An *A-voltage assignment* on Γ is a function $\phi : A(\Gamma) \to A$ such that $\phi(u, v) = \phi(v, u)^{-1}$, for all $(u, v) \in A(\Gamma)$; the values of ϕ are called *voltages*. The graph $\tilde{\Gamma} = \Gamma \times_\phi A$ derived from ϕ is defined by $V(\tilde{\Gamma}) = V(\Gamma) \times A$, and $E(\tilde{\Gamma}) = \{\{(u, g), (v, g\phi(u, v))\}: \{u, v\} \in E(\Gamma), g \in A\}$. The projection onto the first coordinate $\pi : \tilde{\Gamma} = \Gamma \times_\phi A \to \Gamma$ defines an A-covering.

Given a spanning tree T of Γ, a *cotree* edge is one not belonging to T. A voltage assignment ϕ is called *T-reduced* if the voltage associated to each tree arc is the identity. It was shown in [2] that every regular covering of Γ is isomorphic to a derived graph arising from a T-reduced voltage assignment with respect to an arbitrary spanning tree T. The voltage assignment ϕ naturally extends to walks in Γ. For any walk W, let $\phi(W)$ denote the voltage of W.

An automorphism $\alpha \in \mathrm{Aut}(\Gamma)$ is *lifted* to $\tilde{\alpha} \in \mathrm{Aut}(\tilde{\Gamma})$ if $\pi\tilde{\alpha} = \alpha\pi$.

In this paper we consider the following decision problem: Given a graph Γ, a regular covering $\Gamma \times_\phi A$ with A a finite abelian group, and an automorphism α of Γ, decide whether α can be lifted.

The following proposition of [6] plays a key role:

Proposition 1.1. *Let $\tilde{\Gamma} = \Gamma \times_\phi A$ be a regular covering. Then an automorphism α of Γ can be lifted to an automorphism of $\tilde{\Gamma}$ if and only if, for each closed walk W in Γ, one has $\phi(\alpha(W)) = 0$ if $\phi(W) = 0$.*

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Recently, much attention has been paid to the automorphism lifting problem. The motivation was to construct highly transitive graphs; moreover, the lifting conditions are frequently needed in classifying symmetries of certain infinite families of graphs. In [1], a linear criteria is proposed for the case when A is elementary abelian, and applied to classify arc-transitive A-coverings of the Petersen graph. In [8] the authors classify all vertex-transitive elementary abelian coverings of the Petersen graph; the method, developed in [7], reduces the problem to that of finding invariant subspaces of a group representation. See also [5,9], and the references therein.

To the automorphism lifting problem for arbitrary finite abelian groups, we contribute a criterion based on diagonalization of matrices over the ring $\mathbb{Z}/p^k\mathbb{Z}$. The criterion is easy to run in that one only needs to compute some matrices derived from input data, and check whether the entries of the resulting matrices satisfy certain conditions.

Properties of matrices over $\mathbb{Z}/p^k\mathbb{Z}$ that are relevant in the above context, are discussed in Section 2. This is applied in Section 3 in order to derive a criterion for lifting automorphisms. In Section 4, a practical algorithm is presented and illustrated on an example.

2. On matrices over $\mathbb{Z}/p^k\mathbb{Z}$

We start with some notational conventions. Operations in abelian groups are written additively. For a matrix C, denote its (i, j)-entry by $C_{i,j}$, and we often write $C = (C_{i,j})$. For a ring $\mathcal{R}$, let $\mathcal{M}_{m\times n}(\mathcal{R})$ be the set of all $m \times n$-matrices with entries in $\mathcal{R}$. Let $\mathcal{M}_n(\mathcal{R}) = \mathcal{M}_{n\times n}(\mathcal{R})$, and let $\mathrm{GL}(n, \mathcal{R})$ be the set of invertible ones.

Let p be a prime number, and $R = \mathbb{Z}/p^k\mathbb{Z}$. We can speak about the divisibility of an element $\lambda \in R$ by p^r, $0 \leqslant r < k$, without ambiguity. Note that λ is invertible if and only if it is not divisible by p.

Definition 2.1. For $0 \neq \lambda \in R$, the *p-degree* of λ, denoted $d_p(\lambda)$, is the largest integer r, $0 \leqslant r < k$, such that $p^r|\lambda$. By convention set $d_p(0) = k$.

Definition 2.2. A nonzero matrix $X = (X_{i,j}) \in \mathcal{M}_{m\times n}(R)$ is *in normal form* if $X_{i,j} = \delta_{i,j} \cdot p^{r_i}$ for some integers $r_1, \ldots, r_m$, with $0 \leqslant r_1 \leqslant \cdots \leqslant r_l < k = r_{l+1} = \cdots = r_m$ for some $l \leqslant \min\{m, n\}$.

Lemma 2.3. *For each $0 \neq X \in \mathcal{M}_{m\times n}(R)$, there exist $Q \in \mathrm{GL}(m, R)$, $T \in \mathrm{GL}(n, R)$ such that QXT is in normal form.*

Proof. Choose an entry $X_{i_1,j_1} \neq 0$ with smallest p-degree, and suppose $X_{i_1,j_1} = p^{r_1} \cdot \chi$ with χ invertible. Interchange the i_1-th row of X with the first row, and the j_1-th column with the first column, and divide the first row by χ. The matrix obtained has $(1, 1)$-entry p^{r_1}, dividing all entries. Then perform row-transformations to eliminate the $(i, 1)$-entries for $1 < i \leqslant m$. Denote the new matrix by $X^{(1)}$.

Do the same thing to the $(m-1) \times (n-1)$ down-right minor of $X^{(1)}$, and then go on. At each step, we can take row transformations to eliminate elements of a submatrix in a column below the main diagonal, and rearrange the diagonal elements. At the l-th step, for some $l \leqslant \min\{m, n\}$, we get a matrix $X^{(l)}$ such that the i-th diagonal entry is p^{r_i} for $1 \leqslant i \leqslant m$, with $r_1 \leqslant \cdots \leqslant r_m$, and the elements under the main diagonal all vanish.

Finally, perform column transformations to $X^{(l)}$, to eliminate all the "off-diagonal" entries. Then the resulting matrix is in normal form. □

3. The criterion for lifting automorphisms

From now on we make the followings assumptions regarding a connected graph Γ and an A-voltage assignment on Γ.

(1) Let $A = \prod_{\gamma=1}^{g} A_\gamma$ denote an abelian group, with $A_\gamma = \prod_{\eta=1}^{n_\gamma} \mathbb{Z}/p_\gamma^{k(\gamma,\eta)}\mathbb{Z}$, where the prime numbers $p_1, \ldots, p_g$ are distinct, and $k(\gamma, 1) \leqslant \cdots \leqslant k(\gamma, n_\gamma)$ for each γ.
(2) For each edge $e \in E(\Gamma)$, one of its arcs is chosen and denoted by $h(e)$.
(3) A spanning tree T and a vertex $v_0 \in V(T)$ are chosen. For $v_1, v_2 \in V(\Gamma) = V(T)$, let $W(v_1, v_2)$ denote the unique reduced walk in T from v_1 to v_2. Denote by $e_1, \ldots, e_m$ the cotree edges. For each e_i denote by u_i and w_i the head and the tail of $h(e_i)$, respectively. Let L_i be the unique closed reduced walk based at v_0 and containing $h(e_i)$, that is, $L_i = W(v_0, u_i) \cup h(e_i) \cup W(w_i, v_0)$.
(4) Let ϕ be a T-reduced voltage assignment on Γ relative to T and v_0, so $\phi(h(e)) = 0$ for all $e \in E(T)$.

It is well-known that the first homology group $H_1(\Gamma; \mathbb{Z})$ is the free abelian group on the set $\{L_1, \ldots, L_m\}$ (see [4, p. 43]). The voltage assignment ϕ induces a group homomorphism

$$\phi^* : H_1(\Gamma; \mathbb{Z}) \to A, \tag{1}$$

which is surjective by the assumption of connectedness.

For $1 \leqslant \gamma \leqslant g$, let $R_\gamma = \mathbb{Z}/p_\gamma^{k(\gamma,n_\gamma)}\mathbb{Z}$, considered as a ring. Let $R = \mathbb{Z}/r\mathbb{Z}$, with $r = \prod_{\gamma=1}^{g} p_\gamma^{k(\gamma,n_\gamma)}$. There are obvious projections $q_\gamma : R \twoheadrightarrow R_\gamma$ and injections $\iota_\gamma : R_\gamma \hookrightarrow R$. The projections q_γ define the canonical isomorphism $q : R \to \prod_{\gamma=1}^{g} R_\gamma$. Abusing the notation, we denote the induced projections $R^m \twoheadrightarrow R_\gamma^m$, injections $R_\gamma^m \hookrightarrow R^m$ and the isomorphism $R^m \cong \prod_{\gamma=1}^{g} R_\gamma^m$ also by q_γ, ι_γ and q, respectively.