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Note

Congruences for the number of k-tuple partitions with distinct even parts

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ABSTRACT

Let $k \geq 1$ be an integer and $ped_k(n)$ be the number of k-tuple partitions of n wherein even parts are distinct (and odd parts are unrestricted). We prove a class of congruences for $ped_k(n)$ mod 2 by Hecke nilpotency.

1. Introduction

Let $ped(n)$ be the number of partitions of n wherein even parts are distinct (and odd parts are unrestricted). The generating function for $ped(n)$ is

$$\sum_{n=0}^{\infty} ped(n)q^n = \prod_{m=1}^{\infty} \frac{(1+q^{2m})}{(1-q^{2m-1})} = \prod_{m=1}^{\infty} \frac{(1-q^{4m})}{(1-q^m)}. \tag{1}$$

The function $ped(n)$ was studied by Andrews, Hirschhorn and Sellers [2] and some congruence properties were established. For example,

$$ped\left(3^{2\alpha+1}n + \frac{17 \times 3^{2\alpha} - 1}{8}\right) \equiv 0 \pmod 2,$$
$$ped\left(3^{2\alpha+2}n + \frac{19 \times 3^{2\alpha+1} - 1}{8}\right) \equiv 0 \pmod 2$$

for all $\alpha \geq 1$, $n \geq 0$. Such congruences were extended by the author in [3]. (Note that the Ref. [2] is not correctly written in [3].) Indeed, if ℓ is an odd prime and s is an integer satisfying $1 \leq s \leq 8\ell$, $s \equiv 1 \pmod 8$ and $(\frac{s}{\ell}) = -1$, where $(\frac{s}{\ell})$ is the Legendre symbol, then

$$ped\left(\ell^{2\alpha+1}n + \frac{s\ell^{2\alpha} - 1}{8}\right) \equiv 0 \pmod 2.$$

In this note, we consider the following generalization. Let $k \geq 1$ be a positive integer and $ped_k(n)$ be the number of k-tuple partitions of n wherein even parts are distinct. The generating function for $ped_k(n)$ is

$$\sum_{n=0}^{\infty} ped_k(n)q^n = \prod_{m=1}^{\infty} \left(\frac{1+q^{2m}}{1-q^{2m-1}}\right)^k = \prod_{n=1}^{\infty} \frac{(1-q^{4n})^k}{(1-q^n)^k}. \tag{2}$$

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By convention we have $ped_k(n) = 0$ if n is a negative integer. The main result is the following congruences for the function $ped_k(n)$.

Theorem. *Let $k = 2^r s$ is be a positive integer with s odd and $s = \sum_{i=0}^{\infty} \beta_i 2^i$, where $\beta_i = 0$ or 1. Let $g_s = 1 + \sum_{i=0}^{\infty} \beta_{2i+1} 2^i + \sum_{j=0}^{\infty} \beta_{2j+2} 2^j$ and l be an integer such that $l \geq g_s$. Then for any distinct odd primes $\ell_1, \ldots, \ell_l$, we have*

$$ped_k\left(\frac{2^r \ell_1 \ell_2 \cdots \ell_l n - k}{8}\right) \equiv 0 \pmod 2$$

for all n coprime to $\ell_1 \ell_2 \cdots \ell_l$ and satisfying $\ell_1 \ell_2 \cdots \ell_l n \equiv s \pmod 8$.

In particular, for any odd prime ℓ and any integer $\alpha \geq 1$ such that $2^\alpha - 1 \geq g_s$, we have

$$ped_k\left(\frac{2^r \ell^{2^\alpha - 1} n - k}{8}\right) \equiv 0 \pmod 2$$

for all n coprime to ℓ and satisfying $\ell n \equiv s \pmod 8$.

2. Hecke nilpotency

The proof of our theorem relies on Hecke nilpotency of modular forms. We recall some facts on modular forms (see [4]). For integer $k \geq 0$, let M_k (resp., S_k) be the complex vector space of weight k holomorphic modular (resp., cusp) forms with respect to $SL_2(\mathbb{Z})$. Ramanujan's Δ function is

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n \in S_{12},$$

here and throughout $q = e^{2\pi i z}$, and z is on the upper half of the complex plane. Let $\ell \geq 3$ be a prime. If $f(z) = \sum_{n=0}^{\infty} a(n) q^n \in M_k \cap \mathbb{Z}[[q]]$, then the action of the Hecke operator $T_{\ell,k}$ on $f(z) \pmod 2$ is defined by

$$f(z) | T_{\ell,k} = \sum_{n=0}^{\infty} c(n) q^n,$$

where

$$c(n) \equiv \begin{cases} a(\ell n) + a(n/\ell) \pmod 2, & \text{if } \ell | n; \\ a(\ell n) \pmod 2, & \text{if } \ell \nmid n. \end{cases} \tag{3}$$

Since $T_{\ell,k}$ is independent on the weight k of $f(z) \pmod 2$, we abbreviate $T_{\ell,k}$ as T_ℓ for convenience. Based on the work of Serre and Tate (see p. 115 of [6], p. 251 of [7], and [8]), it is known that the action of Hecke algebras on the spaces of modular forms modulo 2 is locally nilpotent. This implies that if $f(z) \in M_k \cap \mathbb{Z}[[q]]$, then there is a positive integer i with the property that

$$f(z) | T_{\ell_1} | T_{\ell_2} \cdots | T_{\ell_i} \equiv 0 \pmod 2$$

for every collection of odd primes $\ell_1, \ell_2, \ldots, \ell_i$. Suppose that $f(z) \not\equiv 0 \pmod 2$. We say $f(z)$ has *degree of nilpotency i* if there exist odd primes $\ell_1, \ell_2, \ldots, \ell_{i-1}$ for which

$$f(z) | T_{\ell_1} | T_{\ell_2} \cdots | T_{\ell_{i-1}} \not\equiv 0 \pmod 2$$

and for every collection of odd primes $p_1, p_2, \ldots, p_i$,

$$f(z) | T_{p_1} | T_{p_2} \cdots | T_{p_i} \equiv 0 \pmod 2.$$

For example, $\Delta(z)$ has degree of nilpotency 1. This is because

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \equiv \sum_{n=1}^{\infty} q^{(2n-1)^2} \pmod 2$$

by Jacobi's triple product (Theorem 2.8 of [1]). This implies that

$$\tau(\ell) \equiv 0 \pmod 2$$

for all primes ℓ. Since $\Delta(z)$ is a Hecke eigenform (p. 164 of [4]), we get

$$\Delta(z) | T_{\ell,12} = \tau(\ell) \Delta(z) \equiv 0 \pmod 2.$$

We denote by g_k the degree of nilpotency of $\Delta^k(z)$. To obtain congruences for $ped_k(n)$, we need the upper bound for g_k. In this direction, Nicolas and Serre (see Théorème 5.1 of [5]) proved the following

Theorem (*Nicolas and Serre*). *For any odd positive integer $k = \sum_{i=0}^{\infty} \beta_i 2^i$, where $\beta_i = 0$ or 1, we have*

$$g_k = 1 + \sum_{i=0}^{\infty} \beta_{2i+1} 2^i + \sum_{j=0}^{\infty} \beta_{2j+2} 2^j.$$