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Gromov hyperbolic graphs

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ABSTRACT

In this paper we prove that the study of the hyperbolicity on graphs can be reduced to the study of the hyperbolicity on simpler graphs. In particular, we prove that the study of the hyperbolicity on a graph with loops and multiple edges can be reduced to the study of the hyperbolicity in the same graph without its loops and multiple edges; we also prove that the study of the hyperbolicity on an arbitrary graph is equivalent to the study of the hyperbolicity on a 3-regular graph obtained from it by adding some edges and vertices. Moreover, we study how the hyperbolicity constant of a graph changes upon adding or deleting finitely or infinitely many edges.

1. Introduction

The study of mathematical properties of Gromov hyperbolic spaces and their applications is a topic of recent and increasing interest in graph theory; see, for instance, [2,3,6–9,14–16,18,19,21–24,26–29].

The theory of Gromov's spaces was used initially for the study of finitely generated groups, where it was demonstrated to have a practical importance. This theory was applied principally in the study of automatic groups (see [20]), that play a role in sciences of computation. Another important application of these spaces is in secure transmission of information on the internet (see [14–16]). Furthermore, the hyperbolicity plays an important role in the spread of viruses through the network (see [14,16]). The hyperbolicity is also useful in the study of DNA data (see [6]).

In recent years several researchers have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Gehring–Osgood j-metric is Gromov hyperbolic; and the Vuorinen j-metric is not Gromov hyperbolic except in a punctured space (see [11]). The study of Gromov hyperbolicity in Riemann surfaces with Poincaré metrics is the subject of [1,4,12,13,22,24–26]. In [5, Section 1.3] it is observed that the hyperbolicity of a geodesic metric space is equivalent to the hyperbolicity of a graph related to it (see also [22,24,26,28] and Theorem 29 in this paper). Hence, establishing hyperbolicity criteria for graphs will be of interest to us.

In our study on hyperbolic graphs we use the notation of [10]. Let (X, d) be a metric space and let $\gamma : [a, b] \longrightarrow X$ be a continuous function. We say that γ is a *geodesic* if $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$, where L denotes the length of a curve. We say that X is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining x and y; we denote by $[xy]$ any such geodesic (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected.

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In order to consider a graph G as a geodesic metric space, if we use the notation uv for the edge joining the vertices u and v, we must identify any edge $uv \in E(G)$ with the real interval $[0, l]$ (if $l := L(uv)$); therefore, any point in the interior of any edge is a point of G and, if we consider the edge uv as a graph with just one edge, then it is isometric to $[0, l]$. A connected graph G is naturally equipped with a distance defined on its points, induced by taking shortest paths in G. Then, we see G as a metric graph.

Throughout the paper we allow loops and multiple edges in the graphs; we also allow edges of arbitrary lengths. We always consider graphs which are connected and locally finite (i.e., in each ball there are just a finite number of edges). These properties guarantee that the graphs are geodesic metric spaces.

If X is a geodesic metric space and $J = \{J_1, J_2, \ldots, J_n\}$ is a polygon, with sides $J_j \subseteq X$, we say that J is δ-*thin* if for every $x \in J_i$ we have that $d(x, \cup_{j\neq i} J_j) \leq \delta$. We denote by $\delta(J)$ the sharp thin constant of J, i.e., $\delta(J) := \inf\{\delta \geq 0 : J \text{ is } \delta\text{-thin}\}$. If $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$. The space X is δ-*hyperbolic* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of X, i.e., $\delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X\}$. We say that X is *hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$. If X is hyperbolic, then $\delta(X) = \inf\{\delta \geq 0 : X \text{ is } \delta\text{-hyperbolic}\}$.

There are several definitions of Gromov hyperbolicity (see, e.g., [5,10]). These different definitions are equivalent in the sense that if X is δ_A-hyperbolic with respect to the definition A, then it is δ_B-hyperbolic with respect to the definition B for some constant δ_B. However, for a fixed $\delta \geq 0$, the set of δ-hyperbolic graphs with respect to the definition A is different, in general, from the set of δ-hyperbolic graphs with respect to the definition B. We have chosen this definition since it has a deep geometric meaning (see, e.g., [10]).

Some authors (see, e.g., [6]) study Gromov hyperbolicity for graphs G such that every edge has length 1; in this context, they define $\delta(G)$ as

$$\sup\{\delta(T) : T \text{ is a geodesic triangle in } G \text{ with vertices in } V(G)\}.$$

This definition is equivalent to our definition if every edge in G has length 1. However, if we want to deal with graphs with edges of arbitrary lengths, we must consider geodesic triangles with vertices in G.

The following are interesting examples of hyperbolic spaces. The real line $\mathbb{R}$ is 0-hyperbolic: in fact, any point of a geodesic triangle in the real line belongs to two sides of the triangle simultaneously, and therefore we can conclude that $\mathbb{R}$ is 0-hyperbolic. The Euclidean plane $\mathbb{R}^2$ is not hyperbolic: it is clear that equilateral triangles can be drawn with arbitrarily large diameter, so $\mathbb{R}^2$ with the Euclidean metric is not hyperbolic. This argument also proves that a normed vector space is hyperbolic if and only it has dimension 1. Every arbitrary length metric tree is 0-hyperbolic: in fact, all points of a geodesic triangle in a tree belong simultaneously to two sides of the triangle.

Those spaces X with $\delta(X) = 0$ are precisely the metric trees, and the hyperbolicity constant of a geodesic metric space can be viewed as a measure of how "tree-like" the space is. Every bounded metric space X is $(\operatorname{diam} X)$-hyperbolic. Every simply connected complete Riemannian manifold with sectional curvature verifying $K \leq -c^2 < 0$ is hyperbolic. See [5,10] for more background and further results.

We would like to point out that deciding whether or not a space is hyperbolic is usually very difficult. Notice that, first of all, we have to consider an arbitrary geodesic triangle T, and compute the minimum distance from an arbitrary point P of T to the union of the other two sides of the triangle to which P does not belong. And then we have to take the supremum over all the possible choices for P and then over all the possible choices for T. This means that if our space is, for instance, an n-dimensional manifold and we select two points P and Q on different sides of a triangle T, the function F that measures the distance between P and Q is a $(3n+2)$-variable function ($3n$ variables describe the three vertices of T and two variables describe the points P and Q in the closed curve given by T). In order to prove that our space is hyperbolic we would have to take the minimum of F over the variable that describes Q, and then the supremum over the remaining $3n + 1$ variables, or at least to prove that it is finite. Without disregarding the difficulty of solving a $(3n + 2)$-variable minimax problem, notice that the main obstacle is that we do not even know in an approximate way the location of the geodesics in the space.

The study of hyperbolic graphs is an interesting topic since, as we have seen, the hyperbolicity of many geodesic metric spaces is equivalent to the hyperbolicity of some graphs related to them. The main aim of this paper is to show that it suffices to study the hyperbolicity of graphs of a very special kind: 3-regular graphs without loops and multiple edges (see Theorems 6, 8 and 19). One way to get around the appearance of loops and multiple edges is just to insert new points in the middle of relevant edges and loops, obtaining a graph with more vertices than the original one; however, we choose a more natural and difficult approach since Theorem 6 (respectively, Theorem 8) relates the hyperbolicity constant of a graph with the hyperbolicity constant of the graph obtained by deleting its loops (respectively, obtained by replacing each multiple edge by just one edge).

2. Hyperbolicity, loops and multiple edges

A *loop* is an edge that connects a vertex to itself and a *multiple edge* is the set of all edges (at least two) which are incident to the same two vertices. We prove in this section that, in order to study Gromov hyperbolicity, it suffices to consider graphs without loops and multiple edges.