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Note

PPAD-completeness of polyhedral versions of Sperner's Lemma

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A B S T R A C T

We show that certain polyhedral versions of Sperner's Lemma, where the colouring is given explicitly as part of the input, are PPAD-complete. The proofs are based on two recent results on the complexity of computational problems in game theory: the PPAD-completeness of 2-player Nash, proved by Chen and Deng, and of Scarf's Lemma, proved by Kintali. We give a strengthening of the latter result, show how colourings of polyhedra provide a link between the two, and discuss a special case related to vertex covers.

1. Introduction

Sperner's Lemma on the existence of a panchromatic triangle in a suitable colouring of a triangulation has many versions and generalizations. The following is a variant formulated in terms of colourings of n-dimensional polytopes, which can be seen as the usual multidimensional Sperner Lemma applied to a subdivision of the Schlegel diagram of a polytope; see [8] for a direct proof. Given a colouring of the vertices of a polytope by n colours, a facet is called *panchromatic* if it contains vertices of each colour.

Theorem 1. *Let P be an n-dimensional polytope, with a simplex facet F_0. Suppose we have a colouring of the vertices of P by n colours such that F_0 is panchromatic. Then there is another panchromatic facet.*

This leads to the problem of finding a panchromatic facet other than F_0.

Polytopal Sperner

Input: vectors $v^i \in \mathbb{Q}^n$ ($i = 1, \dots, m$) whose convex hull is a full-dimensional polytope P; a colouring of the vertices by n colours; a panchromatic simplex facet F_0 of P.
Output: vectors $v^{i_1}, \dots, v^{i_n}$ with different colours which define a facet of P different from F_0.

By the *polar* of a polyhedron P we mean the polyhedron $P^{\Delta} := \{c \in \mathbb{R}^n : cx \le 1 \text{ for all } x \in P\}$. By taking the polar after translating the polytope so that the origin lies in its interior, we get the following polar version of Theorem 1. A vertex of an n-dimensional polyhedron is *simple* if it lies on exactly n facets. For a colouring of the facets, a vertex is *panchromatic* if it lies on facets of every colour.

Theorem 2. *Let P be an n-dimensional polytope, with a simple vertex v_0. Suppose we have a colouring of the facets of P with n colours such that v_0 is panchromatic. Then there is another panchromatic vertex.*

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The corresponding computational problem, where the polytope is given by a linear inequality system, is equivalent with POLYTOPAL SPERNER.

A related but slightly different Sperner-like theorem was introduced by the authors in [8]. Recall that the *extreme directions* of a polyhedron are the extreme rays of its characteristic cone. For a vector v, by *direction* v we mean the direction $\{\lambda v : \lambda \geq 0\}$.

Theorem 3 *([8]). Let P be an n-dimensional pointed polyhedron whose characteristic cone is generated by n linearly independent vectors. If we colour the facets of the polyhedron by n colours such that facets having the i-th extreme direction do not get colour i, then there is a panchromatic vertex.*

The advantage of this theorem over the standard Sperner Lemma is that it enables short and graphic proofs of several combinatorial and game theoretic results about stable sets and matchings; see [9,8,11]. An example of this is the proof of Theorem 9, a new result about vertex covers of graphs. The computational problem corresponding to Theorem 3 is the following.

EXTREME DIRECTION SPERNER

Input: matrix $A \in \mathbb{Q}^{m\times n}$ and vector $b \in \mathbb{Q}^m$ such that $P = \{x : Ax \leq b\}$ is a pointed polyhedron whose characteristic cone is generated by n linearly independent vectors; a colouring of the facets by n colours such that facets having the i-th extreme direction do not get colour i.
Output: a panchromatic vertex of P.

In this note we show, using recent developments on the computational complexity of problems in game theory, that the following two natural special cases of this problem are already PPAD-complete.

0–1 EXTREME DIRECTION SPERNER

Input: matrix $A \in \{0, 1\}^{m\times n}$ with no all-0 column; a colouring of the facets of $P = \{x : Ax \leq \mathbf{1}, x \leq \mathbf{1}\}$ by n colours such that facets with extreme direction $-e_i$ do not get colour i.
Output: a panchromatic vertex of P.

EXTREME DIRECTION SPERNER WITH $2n$ FACETS

Input: a matrix $A \in \mathbb{Q}_+^{n\times n}$; a colouring of the facets of $P = \{x : Ax \leq \mathbf{1}, x \leq \mathbf{1}\}$ by n colours such that facets with extreme direction $-e_i$ do not get colour i and every colour appears exactly twice.
Output: a panchromatic vertex of P.

We also show that EXTREME DIRECTION SPERNER provides a link between the complexity of Scarf's Lemma and that of finding Nash equilibria in 2-player games. In particular, EXTREME DIRECTION SPERNER WITH $2n$ FACETS can be considered as a special case of the computational version of Scarf's Lemma.

The structure of the paper is as follows. The remaining part of this section introduces the complexity class PPAD. In Section 2 we show that our problems belong to this class. Then in Section 3 we use the results of Kintali [6] to show that 0–1 EXTREME DIRECTION SPERNER is PPAD-complete even in the case when each row of A contains at most three 1 s. In contrast, the problem is solvable in polynomial time if each row contains at most two 1 s. If arbitrary left sides are allowed, then we obtain Theorem 9 on vertex covers. Finally, in Section 4 we prove using the result of Chen and Deng [2,3] that EXTREME DIRECTION SPERNER WITH $2n$ FACETS is PPAD-complete. We also show that this problem is in fact a special case of SCARF.

1.1. The class PPAD and PPAD-completeness

The complexity class PPAD is defined as the set of total search problems which are Karp-reducible to the following prototypical problem:

END OF THE LINE

Input: an algorithm that describes a directed graph on $\{0, 1\}^n$, with running time polynomial in n. The digraph has in- and out-degrees at most 1, and $\mathbf{0}$ has in-degree 0 and out-degree 1. The algorithm outputs the out-neighbour and in-neighbour of a given node.
Output: any node in $\{0, 1\}^n \setminus \{\mathbf{0}\}$ that has degree 1 (where the degree is the in-degree plus the out-degree).

A problem in PPAD is called PPAD-*complete* if every other problem in PPAD is Karp-reducible to it. The class PPAD was introduced by Papadimitriou [12], who proved among other results that a computational version of 3D Sperner's Lemma is PPAD-complete. Later Chen and Deng [1] proved that the 2-dimensional problem is also PPAD-complete. The input of these computational versions is the description of a polynomial algorithm that computes a legal colouring, while the number of vertices to be coloured is exponential in the input size. This is conceptually different from the computational problems that we consider, where the input explicitly contains the vertices or facets of a polyhedron and their colouring. Our problems are solvable in polynomial time in fixed dimension since then the number of facets and vertices is polynomial.

For a long time it had been open to find natural PPAD-complete problems that do not have a description of a Turing machine in their input. In 2006, Daskalakis, Goldberg and Papadimitriou [4] proved that approximating Nash-equilibria in