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Completing partial Latin squares with one filled row, column and symbol

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ABSTRACT

Let P be an $n \times n$ partial Latin square every non-empty cell of which lies in a fixed row r, a fixed column c or contains a fixed symbol s. Assume further that s is the symbol of cell (r, c) in P. We prove that P is completable to a Latin square if $n \geq 8$ and n is divisible by 4, or $n \leq 7$ and $n \notin \{3, 4, 5\}$. Moreover, we present a polynomial algorithm for the completion of such a partial Latin square.

1. Introduction

Throughout this paper, n is assumed to be a positive integer. Consider an $n \times n$ array P where each cell contains at most one symbol from $\{1, \ldots, n\}$. P is called a *partial Latin square* if each symbol occurs at most once in every row and column. If no cell in P is empty, then it is a *Latin square*.

The cell in position (i, j) in an array A is denoted by $(i, j)_A$, and the symbol in cell $(i, j)_A$ is denoted by $A(i, j)$. If cell $(i, j)_A$ is empty, then we write $A(i, j) = \emptyset$; and if $A(i, j) = k$, then we say that k is an *entry* of cell $(i, j)_A$. An $r \times s$ array with entries from $\{1, \ldots, n\}$ is called a *Latin rectangle* if each symbol occurs at most once in every row and column, and each cell has precisely one entry.

An $n \times n$ Latin square L is a *completion* of an $n \times n$ partial Latin square P if $L(i, j) = P(i, j)$ for each non-empty cell $(i, j)_P$ of P. P is *completable* if there is such a Latin square. Otherwise, P is *non-completable*. The problem of completing partial Latin squares is a classic within combinatorics and there is a wealth of results in the literature. Let us just mention a few here.

In general, it is an *NP*-complete problem to determine if a partial Latin square is completable [3]. Thus it is natural to ask if particular families of partial Latin squares are completable. A classic result due to Ryser [8] states that if $n \geq r, s$, then every $n \times n$ partial Latin square whose non-empty cells lie in an $r \times s$ Latin rectangle Q is completable if and only if each of the symbols $1, \ldots, n$ occurs at least $r + s - n$ times in Q. Another classic result is Smetaniuk's proof [9] of Evans' conjecture [7] that every $n \times n$ partial Latin square with at most $n - 1$ entries is completable. This was also independently proved by Andersen and Hilton [1]. Let us also mention the following conjecture of Häggkvist.

Conjecture 1.1. *Any $nr \times nr$ partial Latin square whose non-empty cells lie in $(n - 1)$ disjoint $r \times r$ squares can be completed.*

In [5] Conjecture 1.1 was proved for the case $r = 3$. Some other cases were resolved in [4,6]. See also [2] for related results.

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1			
	1		
		1	
			2

Fig. 1. A non-completable partial Latin square with entries only on the main diagonal.

1	2	3
2	1	
3		1

1	3	4	2
2	1		
3		1	
4			1

1	3	2	4	5
2	1			
3		1		
4			1	
5				1

Fig. 2. Non-completable partial Latin squares of order 3, 4 and 5.

In this paper we are interested in completions of a particular family of partial Latin squares. We propose the following conjecture.

Conjecture 1.2. *Let $r, c, s \in \{1, \dots, n\}$, and let P be an $n \times n$ partial Latin square where each non-empty cell lies in row r or column c, or has entry s, and assume further that $P(r, c) = s$. If $n \notin \{3, 4, 5\}$, then P is completable.*

Note that the condition $P(r, c) = s$ is necessary, since removing it yields a conjecture with obvious counterexamples such as partial Latin squares of the form as in Fig. 1.

The non-completable partial Latin squares in Fig. 2 show that the condition $n \notin \{3, 4, 5\}$ in Conjecture 1.2 is necessary.

In this paper we present the following theorem which provides some evidence for Conjecture 1.2.

Theorem 1.3. *Let $r, c, s \in \{1, \dots, n\}$, and let P be an $n \times n$ partial Latin square every non-empty cell of which is in row r or column c, or has entry s, and assume further that $P(r, c) = s$. If $n \notin \{3, 4, 5\}$ and $n \leq 7$, or $n = 4k$ for some integer $k \geq 2$, then P is completable.*

In the remaining part of the paper we prove this theorem. We also note the following slightly more general result that follows from Theorem 1.3.

Corollary 1.4. *Let $r, c, s \in \{1, \dots, n\}$, and let P be an $n \times n$ partial Latin square every non-empty cell of which is in row r or column c, or has entry s. Assume further that $(r, c)_P$ is non-empty and that symbol s appears in row r and column c. If $n \notin \{3, 4, 5\}$ and $n \leq 7$, or $n = 4k$ for some integer $k \geq 2$, then P is completable.*

Proof. Suppose that $P(r, c) \neq s$, since otherwise the result follows immediately from Theorem 1.3. Note also that by a simple application of Hall's theorem, we may assume that each cell in row r and column c is non-empty, and that there are n cells with entry s in P. The case when $n \in \{1, 2\}$ is trivial, so suppose that $n \geq 6$. Let r', c' be integers such that $P(r', c) = s$ and that $P(r, c') = s$. Define a new $n \times n$ partial Latin square R from P by for each integer $i \in \{1, \dots, n\} \setminus \{c, c'\}$, setting $R(r, i) = \emptyset$ and $R(r', i) = P(r, i)$, $R(r', c') = P(r, c)$, and retaining the entry (or non-entry) of every other cell in P. It is easy to see that R satisfies the hypothesis of Theorem 1.3, and thus is completable to a Latin square L. From L we define the Latin square L' by for each integer $i \in \{1, \dots, n\} \setminus \{c, c'\}$, setting $L'(r, i) = L(r', i)$ and $L'(r', i) = L(r, i)$, and retaining the entry of every other cell in L. L' is a completion of P. □

2. Preliminaries

Two partial Latin squares L and L' are *isotopic* if L' can be obtained from L by permuting rows, permuting columns and/or permuting symbols in L. Note that if L and L' are isotopic, then L is completable if and only if L' is completable.

A 2-*square* in a partial Latin square L is a set

$$S = \{(i_1, j_1)_L, (i_1, j_2)_L, (i_2, j_1)_L, (i_2, j_2)_L\}$$

of cells in L such that

$$L(i_1, j_1) = L(i_2, j_2) \quad \text{and} \quad L(i_1, j_2) = L(i_2, j_1)$$

(where possibly $L(i_1, j_1) = \emptyset$ or $L(i_1, j_2) = \emptyset$, i.e. $(i_1, j_1)_L$ and $(i_2, j_2)_L$ or $(i_1, j_2)_L$ and $(i_2, j_1)_L$ might be empty). Note that a 2-square is uniquely determined by two cells in L, if at least one of those cells is non-empty. An (s_1, s_2)-*factor* in L is a non-empty set S of $2k$ cells, where k is a positive integer, satisfying the following conditions:

(i) each row and column in L contains either two or no cells from S,
(ii) each cell in S has entry s_1 or s_2.