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Critical properties of graphs of bounded clique-width[☆]

Vadim V. Lozin[a], Martin Milanič[b,c,*]

[a] DIMAP and Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK
[b] University of Primorska, UP IAM, Muzejski trg 2, SI6000 Koper, Slovenia
[c] University of Primorska, UP FAMNIT, Glagoljaška 8, SI6000 Koper, Slovenia

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ABSTRACT

A graph property is a set of graphs closed under isomorphism. Clique-width is a graph parameter which is important in theoretical computer science because many algorithmic problems that are generally NP-hard admit polynomial-time solutions when restricted to graphs of bounded clique-width. Over the last few years, many properties of graphs have been shown to be of bounded clique-width; for many others, it has been shown that the clique-width is unbounded. The goal of the present paper is to tighten the gap between properties of bounded and unbounded clique-width. To this end, we identify new necessary and sufficient conditions for clique-width to be bounded.

1. Introduction

Clique-width is a relatively young notion generalizing another important graph parameter, *tree-width*, studied in the literature for decades. The notion of clique-width generalizes that of tree-width in the sense that graphs of bounded tree-width have bounded clique-width. The importance of these graph invariants is due to the fact that numerous problems that are generally NP-hard admit polynomial-time solutions when restricted to graphs of bounded tree- or clique-width (see, e.g., [10,14–16,18,21,32]).

The notion of clique-width was introduced in the early 1990s [9], and since then many questions regarding this graph parameter have been addressed, both from theoretical and algorithmic points of view. How to compute the clique-width of a graph? How difficult is this problem? How to recognize graphs of clique-width at most k? Is the clique-width of graphs in a certain class bounded or not?

The question about the complexity of computing the clique-width was settled only recently. In [13], Fellows et al. showed that the problem *"Given a graph G and an integer k, is the clique-width of G at most k?"* is NP-complete. For specific values of k, polynomial-time algorithms have so far been found only for $k \leq 3$ [8], while for higher values the complexity remains unknown.

It is therefore desirable to identify classes of graphs with bounded clique-width. In the study of the notion of *tree-width*, one can be restricted, without loss of generality, to graph classes that are closed under taking minors, since the tree-width of a graph is never smaller than the tree-width of any of its minors. According to the celebrated result of Robertson and Seymour [34] a minor-closed class X is of bounded tree-width if and only if X excludes (i.e., does not contain) at least one planar graph. In other words, in the family of minor-closed graph classes the planar graphs constitute the unique minimal class of unbounded tree-width.

☆ This paper combines and extends some of the results presented at ISAAC 2002, the 13-th International Symposium Algorithms and Computation, and WG 2007, the 33-rd International Workshop on Graph-Theoretic Concepts in Computer Science. Extended abstracts of these presentations appeared in [3] and [26], respectively.

* Corresponding author at: University of Primorska, UP IAM, Muzejski trg 2, SI6000 Koper, Slovenia.
E-mail addresses: V.Lozin@warwick.ac.uk (V.V. Lozin), martin.milanic@upr.si (M. Milanič).

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No such criterion is known for the notion of *clique-width*, and the situation with clique-width is more complicated. One problem is that in the case of clique-width, considering minor-closed classes of graphs does not give the desired result, since the clique-width of a graph can be (much) smaller than the clique-width of some minor of it. However, the clique-width of a graph can never be less than the clique-width of any of its induced subgraphs [11], which allows us to restrict ourselves to hereditary classes, i.e., those containing with every graph G all induced subgraphs of G. The family of hereditary graph classes extends that of minor-closed classes and contains many other classes of theoretical or practical importance, such as perfect, bipartite, interval graphs, etc. Over the last few years, many hereditary classes of graphs have been shown to be of bounded clique-width; for many others, it has been shown that the clique-width is unbounded (see, e.g., [3–7,17,19,20,22,23,25,27–31,33,36]).

By analogy with the characterization of the family of minor-closed graph classes of bounded tree-width by means of the unique minimal minor-closed class of graphs of unbounded tree-width, we would like to characterize the family of hereditary classes with bounded clique-width in terms of minimal hereditary classes of unbounded clique-width. To some extent, this approach works, and the first two minimal hereditary classes of unbounded clique-width have been identified recently in [25]. However, there is a fundamental difference between minor-closed and hereditary classes which makes the above analogy generally impossible: there exist hereditary classes of unbounded clique-width containing no minimal class of unbounded clique-width. This is because the induced subgraph relation is not a well-quasi-order, i.e., it contains infinite antichains (sets of graphs pairwise incomparable with respect to the relation), while the minor relation is a well-quasi-order [35].

Two simple examples of infinite antichains with respect to the induced subgraph relation are cycles $C_3, C_4, C_5, \ldots$ and graphs of the form H_i represented in Fig. 1. In [29], it was proved that for any fixed $k \geq 3$, the clique-width is unbounded in the class of graphs not containing any C_i or H_i with $i \leq k$ as an induced subgraph. On the other hand, in the limit class of this sequence (i.e., with $k \to \infty$) the clique-width is bounded. In Section 3 of the present paper, we elaborate on this example and tighten the restrictions under which the clique-width remains unbounded. Then in Section 4 we turn to classes where the restrictions fail. By examining different ways of violating these restrictions, we reveal several families of monotone classes and of classes of bounded degree that are of bounded clique-width.

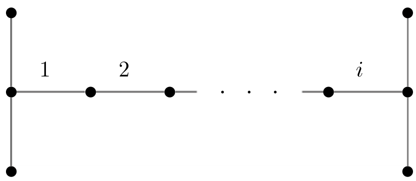

Fig. 1. Graphs H_i.

2. Definitions and notations

All graphs in this paper are finite, undirected, without loops and multiple edges. The vertex set and the edge set of a graph G are denoted $V(G)$ and $E(G)$, respectively. As usual, C_n and P_n stand for the chordless cycle and the chordless path on n vertices, respectively. Also, $K_{n,m}$ is the complete bipartite graph with parts of size n and m.

The *clique-width* of a graph G is the minimum number of labels needed to construct G using the following four operations:

(i) Creation of a new vertex v with label i (the operation is denoted by $i(v)$).
(ii) Disjoint union of two labeled graphs G and H (denoted by $G \oplus H$).
(iii) Joining by an edge each vertex with label i to each vertex with label j ($i \neq j$, denoted by $\eta_{i,j}$).
(iv) Renaming label i to j (denoted by $\rho_{i\to j}$).

Every graph can be defined by an algebraic expression using these four operations. For instance, a chordless path on five consecutive vertices a, b, c, d, e can be defined as follows:

$$\eta_{3,2}(3(e) \oplus \rho_{3\to 2}(\rho_{2\to 1}(\eta_{3,2}(3(d) \oplus \rho_{3\to 2}(\rho_{2\to 1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a))))))))).$$

Such an expression is called a *k-expression* if it uses at most k different labels. The clique-width of G, denoted $\mathrm{cw}(G)$, is the minimum k for which there exists a k-expression defining G.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In other words, H is a subgraph of G if H can be obtained from G by vertex deletions and edge deletions. A graph H is an *induced* subgraph of G if H can be obtained from G by vertex deletions alone. For a subset $U \subseteq V(G)$, we denote by $G - U$ the induced subgraph of G obtained by deleting the vertices of U. We also denote the subgraph of G induced by a set W by $G[W]$. A graph H is said to be a *minor* of a graph G if H can be obtained from G by means of vertex deletions, edge deletions and edge contractions. *Contracting an edge uv* in a graph G means replacing the edge uv together with its two endpoints with a new vertex adjacent precisely to all neighbors of either u or v in G.