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The metric dimension of the lexicographic product of graphs

S.W. Saputro [a,*], R. Simanjuntak [a], S. Uttunggadewa [a], H. Assiyatun [a], E.T. Baskoro [a], A.N.M. Salman [a], M. Bača [b]

[a] Combinatorial Mathematics Research Division, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jl. Ganesha 10 Bandung 40132, Indonesia
[b] Department of Appl. Mathematics, Technical University, Letná 9, 042 00 Košice, Slovak Republic

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ABSTRACT

A set of vertices W *resolves* a graph G if every vertex is uniquely determined by its coordinate of distances to the vertices in W. The minimum cardinality of a resolving set of G is called the *metric dimension* of G. In this paper, we consider a graph which is obtained by the lexicographic product between two graphs. The *lexicographic product* of graphs G and H, which is denoted by $G \circ H$, is the graph with vertex set $V(G) \times V(H) = \{(a, v) \,|\, a \in V(G), v \in V(H)\}$, where (a, v) is adjacent to (b, w) whenever $ab \in E(G)$, or $a = b$ and $vw \in E(H)$. We give the general bounds of the metric dimension of a lexicographic product of any connected graph G and an arbitrary graph H. We also show that the bounds are sharp.

1. Introduction

Throughout this paper, all graphs G are finite and simple. We denote by V the vertex set of G and by E the edge set of G. The distance between two vertices $u, v \in V(G)$, denoted by $d(u, v)$, is the length of a shortest $u - v$ path in G. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered subset of $V(G)$. For $v \in V(G)$, a *representation* of v with respect to W is defined as the k-tuple $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. The set W is called a *resolving set* of G if every two distinct vertices $x, y \in V(G)$ satisfy $r(x|W) \neq r(y|W)$. A *basis* of G is a resolving set of G with the minimum cardinality, and the *metric dimension* of G refers to its cardinality and is denoted by $\beta(G)$.

The metric dimension problems were first studied by Harary and Melter [6], and independently by Slater [18,19]. Khuller et al. [11] studied the metric dimension motivated by the robot navigation in a graph space. A resolving set for a graph corresponds to the presence of distinctively labeled "landmark" nodes in the graph. It is assumed that a robot can detect the distance to each node of the landmarks, and hence uniquely determine its location in the graph.

Garey and Johnson [5], and also Khuller et al. [11], showed that determining the metric dimension of an arbitrary graph is an NP-complete problem. However, Chartrand et al. [3] have obtained some results as follows.

Theorem 1 (*[3]*). *Let G be a connected graph of order $n \geq 2$. Then*

1. $\beta(G) = 1$ *if and only if* $G = P_n$.
2. $\beta(G) = n - 1$ *if and only if* $G = K_n$.

* Corresponding author.
E-mail addresses: codesherly_aptx98@yahoo.co.uk, suhadi@math.itb.ac.id (S.W. Saputro), rino@math.itb.ac.id (R. Simanjuntak), s_uttunggadewa@math.itb.ac.id (S. Uttunggadewa), hilda@math.itb.ac.id (H. Assiyatun), ebaskoro@math.itb.ac.id (E.T. Baskoro), msalman@math.itb.ac.id (A.N.M. Salman), Martin.Baca@tuke.sk (M. Bača).

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3. *For* $n \geq 3$, $\beta(C_n) = 2$.
4. $\beta(G) = n - 2$ *if and only if* G *is either* $K_{r,s}$ *for* $r, s \geq 1$, *or* $K_r + \overline{K_s}$ *for* $r \geq 1, s \geq 2$, *or* $K_r + (K_1 \cup K_s)$ *for* $r, s \geq 1$.

Many researchers have also considered this problem for certain particular classes of graphs, such as trees [3,6,11], fans [2], wheels [1,2,17], complete n-partite graphs [3,16], unicyclic graphs [14], grids [13], honeycomb networks [12], circulant networks [15], Cayley graphs [4], graphs with pendants [9], amalgamation of cycles [10], and Jahangir graphs [20].

There are also some results of the metric dimension problem for graphs resulting from operations on graphs. We recall that the *joint graph* of G and H, which is denoted by $G + H$, is a graph with $V(G+H) = V(G) \cup V(H)$ with $V(G) \cap V(H) = \emptyset$ and $E(G+H) = E(G) \cup E(H) \cup \{xy | x \in V(G), y \in V(H)\}$. Some results on certain joint product graphs have been proved in [1,2,17].

Caceres et al. [2], Khuller et al. [11], and Melter et al. [13] have determined the metric dimension of graphs which are obtained by the Cartesian product of two or more graphs. Some graphs which are constructed by the corona product of two graphs have been studied in [9,8,21]. In this paper, we study the metric dimension of the *lexicographic product* of connected graph G and an arbitrary graph H. We give general bounds of the metric dimension and also show that the bounds are sharp.

2. The main results

The *lexicographic product* of graphs G and H, which is denoted by $G \circ H$ [7], is the graph with vertex set $V(G) \times V(H) = \{(a, v) | a \in V(G), v \in V(H)\}$, where (a, v) is adjacent to (b, w) whenever $ab \in E(G)$, or $a = b$ and $vw \in E(H)$. For any vertex $a \in V(G)$ and $b \in V(H)$, we define the vertex set $H(a) = \{(a, v) | v \in V(H)\}$ and $G(b) = \{(v, b) | v \in V(G)\}$.

Let G be a connected graph with $|V(G)| \geq 2$ and H be an arbitrary graph containing k components $H_1, H_2, \ldots, H_k$ and $|V(H)| \geq 2$. For $a \in V(G)$ and $1 \leq i \leq k$, we define the vertex set $H_i(a) = \{(a, v) \mid v \in V(H_i)\}$. We obtain the following propositions.

Proposition 1. *Let* a *and* b *be two distinct vertices in* G. *Every two different vertices* $x, y \in H(a)$ *satisfy* $d(x, z) = d(y, z)$ *whenever* $z \in H(b)$.

Proof. Let $V(H) = \{h_1, h_2, \ldots, h_{|V(H)|}\}$. Let $x = (a, h_p)$, $y = (a, h_q)$, and $z = (b, h_r)$ where $p, q, r \in \{1, 2, \ldots, |V(H)|\}$ and $p \neq q$. Note that, by the definition of $G \circ H$, every vertex of $H(u)$ is adjacent to every vertex of $H(v)$ for $uv \in E(G)$. Now, for $a \in V(G)$, let u_a be a projection of all vertices of $H(a)$. Let Q be a graph where $V(Q) = \{u_a \mid a \in V(G)\}$ and $u_a u_b \in E(Q)$ whenever $ab \in E(G)$. So, the distance between x and z, $d(x, z)$, in $G \circ H$ is equal to the distance between u_a and u_b, $d(u_a, u_b)$, in Q. Since a vertex y is also projected to u_a, we obtain that $d(y, z) = d(u_a, u_b) = d(x, z)$. □

Proposition 2. *For* $a \in V(G)$ *and* $i, j \in \{1, 2, \ldots, k\}$ *with* $i \neq j$, *every two different vertices* $x, y \in H_j(a)$ *satisfy* $d(x, z) = d(y, z)$ *whenever* $z \in H_i(a)$.

Proof. Let $b \in V(G)$ and $ab \in E(G)$. Since all vertices of $H(a)$ are adjacent to all vertices of $H(b)$, for $w \in H(b)$, we obtain that $d(x, z) = d(x, w) + d(w, z) = 2 = d(y, w) + d(w, z) = d(y, z)$. □

By considering Propositions 1 and 2, in order to find a resolving set of $G \circ H$ we must find a subset $S_i(a) \subseteq H_i(a)$ for every $i \in \{1, 2, \ldots, k\}$ and $|V(H_i)| \geq 2$, such that every two distinct vertices $x, y \in H_i(a)$ satisfy $r(x|S_i(a)) \neq r(y|S_i(a))$, which can be seen in the following lemma.

Lemma 1. *Let* G *be a connected graph with* $|V(G)| \geq 2$ *and* H *be an arbitrary graph containing* $k \geq 1$ *components* $H_1, H_2, \ldots, H_k$ *and* $|V(H)| \geq 2$. *Let* W *be a basis of* $G \circ H$. *For any vertex* $a \in V(G)$, *if* $S_i(a) = W \cap H_i(a)$ *for every* $i \in \{1, 2, \ldots, k\}$ *where* $V(H_i) \geq 2$, *then* $S_i(a) \neq \emptyset$. *Moreover, if* B_i *is a basis of* H_i, *then* $|S_i(a)| \geq |B_i|$.

Proof. Suppose that there exists $a \in V(G)$ such that there exists $i \in \{1, 2, \ldots, k\}$ which is satisfying $|V(H_i)| \geq 2$ and $S_i(a) = \emptyset$. Since $|V(H_i)| \geq 2$, by Propositions 1 and 2, there exist two different vertices $(a, x), (a, y) \in H_i(a)$ such that $r((a, x)|W) = r((a, y)|W)$, a contradiction.

Now, suppose that $S_i(a) = \{(a, s_1), (a, s_2), \ldots, (a, s_t)\}$ where $t < |B_i|$ for some basis B_i of H_i. Let us consider $S' = \{s_1, s_2, \ldots, s_t\}$ subset of $V(H_i)$. Since $|S'| < |B_i|$, there exist two distinct vertices $x, y \in V(H_i)$ such that $r(x|S') = r(y|S')$. So, for every $p \in \{1, 2, \ldots, t\}$, we have $d(x, s_p) = d(y, s_p)$. Note that, for every two distinct vertices $u, v \in V(H_i)$, if $d(u, v) \leq 2$ then $d((a, u), (a, v)) = d(u, v)$, otherwise $d((a, u), (a, v)) = 2$. Thus we obtain $d((a, x), (a, s_p)) = d((a, y), (a, s_p))$, and so $r((a, x)|S_i(a)) = r((a, y)|S_i(a))$, a contradiction. □

For a graph H containing singleton components, we obtain the lemma below.

Lemma 2. *Let* G *be a connected graph with* $|V(G)| \geq 2$ *and* H *be an arbitrary graph containing* $k \geq 1$ *components* $H_1, H_2, \ldots, H_k$ *where* $1 \leq |V(H_1)| \leq |V(H_2)| \leq \cdots \leq |V(H_k)|$ *and* $|V(H)| \geq 2$. *Let* W *be a basis of* $G \circ H$. *For any vertex* $a \in V(G)$, *let* $W(a) = W \cap H(a)$. *If* H *contains* $m \geq 1$ *singleton components, then* $W(a)$ *contains at least* $m - 1$ *vertices of* $H_1(a) \cup H_2(a) \cup \cdots \cup H_m(a)$.

Proof. For $m = 1$, let $a \in V(G)$ and $x \in V(H_1)$. Let W be a resolving set of $H_2(a) \cup H_3(a) \cup \cdots \cup H_k(a)$. Note that, for a vertex $u \in V(H_i)$ and $v \in V(H) \setminus V(H_i)$ where $i \in \{1, 2, \ldots, k\}$, $d((a, u), (a, v)) = 2$. So, $r((a, x) \mid W) = (2, 2, \ldots, 2)$.