Discrete Mathematics 312 (2012) 2760–2775

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Permutation patterns and statistics

Theodore Dokos[a,1], Tim Dwyer[b,1], Bryan P. Johnson[c,1], Bruce E. Sagan[c,*,1], Kimberly Selsor[d,1]

[a] Department of Mathematics, The Ohio State University, 100 Math Tower, 231 West 18th Avenue, Columbus, OH 43210-1174, USA
[b] Department of Mathematics, University of Florida, 358 Little Hall, PO Box 118105, Gainesville, FL 32611-8105, USA
[c] Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027, USA
[d] Department of Mathematics, University of South Carolina, LeConte College, 1523 Greene Street, Columbia, SC 29208, USA

A R T I C L E I N F O

Article history:
Received 23 September 2011
Received in revised form 29 April 2012
Accepted 21 May 2012
Available online 13 June 2012

Keywords:
Avoidance
Catalan number
Fibonacci number
Foata bijection
Generating function
Integer partition
Pattern
Permutation
Integer partition
Inversion number
Mahonian pair
Major index
q-analogue
Statistic

A B S T R A C T

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $\{1, 2, \ldots, n\}$ and let $\mathfrak{S} = \cup_{n\geq 0} \mathfrak{S}_n$. If $\Pi \subseteq \mathfrak{S}$ is a set of permutations, then we let $\mathrm{Av}_n(\Pi)$ be the set of permutations in $\mathfrak{S}_n$ which avoid every permutation of Π in the sense of pattern avoidance. One of the celebrated notions in pattern theory is that of Wilf-equivalence, where Π and Π' are *Wilf equivalent* if $\#\mathrm{Av}_n(\Pi) = \#\mathrm{Av}_n(\Pi')$ for all $n \geq 0$. In a recent paper, Sagan and Savage proposed studying a q-analogue of this concept defined as follows. Suppose $\mathrm{st} : \mathfrak{S} \to \{0, 1, 2, \ldots\}$ is a permutation statistic and consider the corresponding generating function $F_n^{\mathrm{st}}(\Pi; q) = \sum_{\sigma\in\mathrm{Av}_n(\Pi)} q^{\mathrm{st}\,\sigma}$. Call Π, Π' *st-Wilf equivalent* if $F_n^{\mathrm{st}}(\Pi; q) = F_n^{\mathrm{st}}(\Pi'; q)$ for all $n \geq 0$. We present the first in-depth study of this concept for the inv and maj statistics. In particular, we determine all inv- and maj-Wilf equivalences for any $\Pi \subseteq \mathfrak{S}_3$. This leads us to consider various q-analogues of the Catalan numbers, Fibonacci numbers, triangular numbers, and powers of two. Our proof techniques use lattice paths, integer partitions, and Foata's second fundamental bijection. We also answer a question about Mahonian pairs raised in the Sagan–Savage article.

1. Introduction

We begin with some well-known definitions. Let $\mathfrak{S}_n$ denote the set of permutations $\pi = a_1a_2 \ldots a_n$ of $[n] = \{1, 2, \ldots, n\}$, and let $\mathfrak{S} = \cup_{n\geq 0} \mathfrak{S}_n$. We denote by $\pi(k)$ the entry a_k. And if $\pi \in \mathfrak{S}_n$ then we say that π has *length* n. Two sequences of distinct integers, $a_1a_2 \ldots a_n$ and $b_1b_2 \ldots b_n$, are said to be *order isomorphic* whenever they satisfy

$$a_i < a_j \quad \text{if and only if} \quad b_i < b_j$$

for all $1 \leq i < j \leq n$. We say that $\sigma \in \mathfrak{S}_n$ *contains a copy of* $\pi \in \mathfrak{S}_k$ *as a pattern* if there is a subsequence of σ order isomporphic to π. For example, $\sigma = 436152$ contains the pattern $\pi = 132$ because of the copy 365. On the other hand, σ *avoids* π if it does not contain π and we let

$$\mathrm{Av}_n(\pi) = \{\sigma \in \mathfrak{S}_n \mid \sigma \text{ avoids } \pi\}.$$

* Corresponding author.
E-mail addresses: t.dokos@gmail.com (T. Dokos), jtimdwyer@gmail.com (T. Dwyer), john2954@msu.edu (B.P. Johnson), sagan@math.msu.edu (B.E. Sagan), selsork@email.sc.edu (K. Selsor).

1 Research partially supported by NSA grant H98230-11-1-0222 and by Michigan State University.

0012-365X/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2012.05.014

More generally, if $\Pi \subseteq \mathfrak{S}$ then we define $\mathrm{Av}_n(\Pi) = \cap_{\pi\in\Pi} \mathrm{Av}_n(\pi)$ so that these permutations avoid every element of Π. Just as with symmetric groups, it will be useful to let

$$\mathrm{Av}(\Pi) = \bigcup_{n\geq 0} \mathrm{Av}_n(\Pi).$$

We say that Π, Π' are *Wilf equivalent* and write $\Pi \equiv \Pi'$ if, for all $n \geq 0$,

$$\#\mathrm{Av}_n(\Pi) = \#\mathrm{Av}_n(\Pi')$$

where the number sign denotes cardinality. Given a specific set Π, we will drop the curly brackets enclosing the set when writing $\mathrm{Av}_n(\Pi)$ and in similar notations below. A famous result in this area states that $\pi \equiv \pi'$ for any $\pi, \pi' \in \mathfrak{S}_3$.

In a recent paper [17], Sagan and Savage proposed studying a q-analogue of Wilf equivalence defined as follows. A *permutation statistic* is a function

$$\mathrm{st} : \mathfrak{S} \to \mathbb{N},$$

where $\mathbb{N}$ is the nonnegative integers. Associated with any statistic st and any $\Pi \subseteq \mathfrak{S}$ we have the generating function

$$F_n^{\mathrm{st}}(\Pi; q) = \sum_{\sigma\in\mathrm{Av}_n(\Pi)} q^{\mathrm{st}\,\sigma}.$$

Call Π, Π' *st-Wilf equivalent*, written $\Pi \overset{\mathrm{st}}{\equiv} \Pi'$, if

$$F_n^{\mathrm{st}}(\Pi; q) = F_n^{\mathrm{st}}(\Pi'; q)$$

for all $n \geq 0$. We denote the st-Wilf equivalence class of Π by $[\Pi]_{\mathrm{st}}$. Note that st-Wilf equivalence implies Wilf equivalence since setting $q = 1$ gives $F_n^{\mathrm{st}}(\Pi; 1) = \#\mathrm{Av}_n(\Pi)$. In particular, if $\pi \overset{\mathrm{st}}{\equiv} \pi'$ then π and π' must be in the same symmetric group since this is true of Wilf equivalence.

Our goal is to study this concept and related ideas for two famous permutation statistics: the inversion number and the major index. The set of *inversions* of $\sigma = a_1 a_2 \ldots a_n$ is

$$\mathrm{Inv}\,\sigma = \{(i, j) \mid i < j \text{ and } a_i > a_j\}.$$

So $\mathrm{Inv}\,\sigma$ records pairs of indices where the corresponding elements in σ are out of order. The *inversion number* of σ is just

$$\mathrm{inv}\,\sigma = \#\mathrm{Inv}\,\sigma.$$

By way of illustration, if $\sigma = 41523$

$$\mathrm{Inv}\,\sigma = \{(1, 2),\ (1, 4),\ (1, 5),\ (3, 4),\ (3, 5)\}$$

since $a_1 = 4 > a_2 = 1$ and so forth, giving $\mathrm{inv}\,\sigma = 5$.

To define the major index, we must first define descents. Permutation $\sigma = a_1 a_2 \ldots a_n$ has *descent set*

$$\mathrm{Des}\,\sigma = \{i \mid a_i > a_{i+1}\}$$

and *descent number*

$$\mathrm{des}\,\sigma = \#\mathrm{Des}\,\sigma.$$

Descents keep track of the first index in inversion pairs consisting of adjacent elements. The *major index* of σ is given by

$$\mathrm{maj}\,\sigma = \sum_{i\in\mathrm{Des}\,\sigma} i.$$

Continuing our example from the previous paragraph, we see that

$$\mathrm{Des}\,\sigma = \{1, 3\}$$

and so $\mathrm{maj}\,\sigma = 1 + 3 = 4$.

The statistics inv and maj are intimately connected and, in fact, are equidistributed over $\mathfrak{S}_n$ in the sense that

$$\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}\,\sigma} = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}\,\sigma}.$$

A statistic whose distribution over $\mathfrak{S}_n$ equals this one is called *Mahonian*. Since we will use them so often, we will adopt the following abbreviations for the generating functions for these statistics over avoidance sets

$$I_n(\Pi; q) = F_n^{\mathrm{inv}}(\Pi; q) \quad \text{and} \quad M_n(\Pi; q) = F_n^{\mathrm{maj}}(\Pi; q).$$

This article is devoted to a study of $I_n(\Pi)$ and $M_n(\Pi)$ and, by the end, we will have completely classified all these polynomials for all $\Pi \subseteq \mathfrak{S}_3$ just as Simion and Schmidt [18] did with the corresponding cardinalities when $q = 1$. Along the way we will meet a number of interesting q-analogues for known combinatorial quantities such as Catalan numbers, Fibonacci numbers, triangular numbers, and powers of two. We will establish connections with lattice paths, integer partitions, and Foata's second fundamental bijection. We will also answer a question of Sagan and Savage [17] concerning certain objects called Mahonian pairs. The rest of this paper is structured as follows. In the following section we will consider inv- and maj-Wilf equivalence in the case $\#\Pi = 1$. Section 3 will talk about q-analogues of the Catalan numbers arising in this context. In Sections 4 and 5 we will look at $I_n(\Pi)$ and $M_n(\Pi)$, respectively, when $\Pi \subset \mathfrak{S}_3$ has cardinality 2. The next two sections investigate the analogous problem for $\#\Pi = 3$. The final section contains concluding remarks. Various conjectures are scattered throughout the paper.