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Pairs of forbidden induced subgraphs for homogeneously traceable graphs

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ABSTRACT

A graph G is called homogeneously traceable if for every vertex v of G, G contains a Hamilton path starting from v. For a graph H, we say that G is H-free if G contains no induced subgraph isomorphic to H. For a family $\mathcal{H}$ of graphs, G is called $\mathcal{H}$-free if G is H-free for every $H \in \mathcal{H}$. Determining families of graphs $\mathcal{H}$ such that every $\mathcal{H}$-free graph G has some graph property has been a popular research topic for several decades, especially for Hamiltonian properties, and more recently for properties related to the existence of graph factors. In this paper we give a complete characterization of all pairs of connected graphs R, S such that every 2-connected $\{R, S\}$-free graph is homogeneously traceable.

1. Introduction

We use Bondy and Murty [3] for terminology and notation not defined here and consider finite simple graphs only.

Let G be a graph. If a subgraph G' of G contains all edges $xy \in E(G)$ with $x, y \in V(G')$, then G' is called an *induced subgraph* of G (or a subgraph of G induced by $V(G')$). For a given graph H, we say that G is H-*free* if G does not contain an induced subgraph isomorphic to H. For a family $\mathcal{H}$ of graphs, G is called $\mathcal{H}$-free if G is H-free for every $H \in \mathcal{H}$. Note that if H_1 is an induced subgraph of H_2, then an H_1-free graph is also H_2-free.

The only graph on four vertices with degree sequence 1, 1, 1, 3 is denoted as $K_{1,3}$ and called a *claw*; the vertex with degree 3 is called the *center* of the claw. Instead of $K_{1,3}$-free, we say that a graph is *claw-free* if it does not contain a copy of $K_{1,3}$ as an induced subgraph. For a subgraph H of G, the vertices with degree 1 in H are called its *end vertices*.

Let P_i be the path on $i \geq 1$ vertices, and C_i the cycle on $i \geq 3$ vertices. We use Z_i to denote the graph obtained by identifying a vertex of a C_3 with an end vertex of a P_{i+1} $(i \geq 1)$, $B_{i,j}$ for the graph obtained by identifying two vertices of a C_3 with the end vertices of a P_{i+1} $(i \geq 1)$ and a P_{j+1} $(j \geq 1)$, respectively, and $N_{i,j,k}$ for the graph obtained by identifying the three vertices of a C_3 with the end vertices of a P_{i+1} $(i \geq 1)$, P_{j+1} $(j \geq 1)$ and P_{k+1} $(k \geq 1)$, respectively. In particular, we let $B = B_{1,1}$ (this graph is sometimes called a *bull*) and $N = N_{1,1,1}$ (this graph is sometimes called a *net*). The graphs $B_{1,4}$, $B_{2,3}$ and $N_{1,1,3}$ play a crucial role in the sequel, and are depicted in Fig. 1.

Adopting the terminology of [3], we call a graph G *Hamiltonian* if it contains a *Hamilton cycle*, i.e., a cycle containing all its vertices, *traceable* if it contains a *Hamilton path*, i.e., a path containing all its vertices, and *Hamilton-connected* if for every pair of vertices x, y of G, G contains a Hamilton path starting from x and terminating in y. We say that G is *homogeneously traceable* if for every vertex x of G, G contains a Hamilton path starting from x. Homogeneously traceable graphs have been introduced by Skupień (see, e.g., [10]), but we do not know whether he is the original author of the concept.

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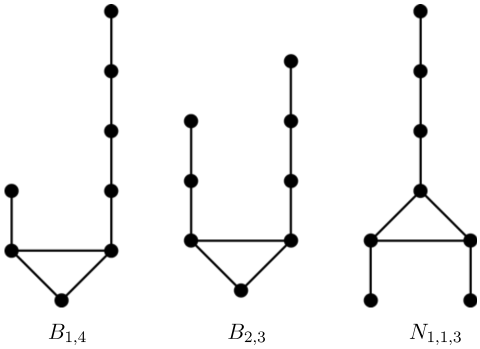

Fig. 1. The graphs $B_{1,4}$, $B_{2,3}$ and $N_{1,1,3}$.

Note that a Hamilton-connected graph (on at least three vertices) is Hamiltonian, that a Hamiltonian graph is homogeneously traceable, and that a homogeneously traceable graph is traceable, but that the reverse statements do not hold in general.

If a graph is connected and P_3-free, then it is a *complete graph*, i.e., its vertex set is a *clique*, i.e., all its vertices are mutually adjacent, and hence it is (homogeneously) traceable, and Hamiltonian if it has order at least 3. In fact, it is not hard to show that the statement 'every connected H-free graph is traceable' only holds if $H = P_3$ (or $H = P_2$, but in that case the statement is trivial). The case with pairs of forbidden subgraphs (different from P_2 and P_3) is much more interesting. For a connected graph to be traceable or Hamiltonian, the following theorem is one of the earliest of this kind.

Theorem 1 (*Duffus et al. [4]*)**.** *Let G be a $\{K_{1,3}, N\}$-free graph.*

(1) *If G is connected, then G is traceable.*
(2) *If G is 2-connected, then G is Hamiltonian.*

Obviously, if H is an induced subgraph of N, then $\{K_{1,3}, H\}$-free instead of $\{K_{1,3}, N\}$-free yields the same conclusions in the above theorem. In particular, if we exclude P_2 as an induced subgraph, we consider graphs without edges, and we obtain trivial statements only. For this reason, throughout we assume that our forbidden subgraphs have at least three vertices. We also assume that our forbidden subgraphs are connected. A natural problem that, as far as we know, was considered for the first time in the Ph.D. Thesis of Bedrossian [2], is to characterize all pairs of forbidden subgraphs for hamiltonicity (and other graph properties). Faudree and Gould [6] later refined this approach by adding a lower bound on the number of vertices of the graph G in order to avoid small, more or less pathological, cases. Restricting our attention to traceability, they proved that (apart from trivial cases) the claw and any of the induced subgraphs of the net are the only forbidden pairs for the property of being traceable.

Theorem 2 (*Faudree and Gould [6]*)**.** *Let R and S be connected graphs with $R, S \neq P_2, P_3$ and let G be a connected graph. Then G being $\{R, S\}$-free implies G is traceable if and only if (up to symmetry) $R = K_{1,3}$ and S is P_4, C_3, Z_1, B or N.*

In the same paper, they discuss analogous results for other Hamiltonian properties. For many of these properties counterparts of Theorem 2 have been established, but for Hamilton-connectedness only partial results are known to date. We refer to [6] for more details. The property of being homogeneously traceable was not addressed in [6] and, as far as we are aware, has not been considered before. Recently, similar questions related to the existence of perfect matchings and 2-factors have been studied. We refer the interested reader to [8,9,1,5,7], respectively, for more details.

In the sequel we solve the analogous problem for homogeneously traceable graphs, so we are going to characterize the pairs of connected forbidden induced subgraphs that imply that a given graph is homogeneously traceable. Note that if a graph contains a cut vertex v, it cannot be homogeneously traceable since there exists no Hamilton path starting at v. So, apart from K_1 and K_2, all homogeneously traceable graphs are 2-connected. Thus we only consider 2-connected graphs. As noted before, if a connected graph G is P_3-free, then it is a complete graph, and hence trivially homogeneously traceable, and in fact it is easy to prove the following statement. We postpone the proof of the 'only-if' part of the next statement to Section 3.

Theorem 3. *Let $S \neq P_2$ be a connected graph and let G be a 2-connected graph. Then G being S-free implies G is homogeneously traceable if and only if $S = P_3$.*

A natural and more interesting problem is to consider pairs of forbidden subgraphs for this property. In this paper, we characterize all such pairs by proving the following result.

Theorem 4. *Let R and S be connected graphs with $R, S \neq P_2, P_3$ and let G be a 2-connected graph. Then G being $\{R, S\}$-free implies G is homogeneously traceable if and only if (up to symmetry) $R = K_{1,3}$ and S is an induced subgraph of $B_{1,4}$, $B_{2,3}$ or $N_{1,1,3}$.*