Discrete Mathematics 312 (2012) 2832–2842

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Families of small regular graphs of girth 5

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ARTICLE INFO

Article history:
Received 8 June 2011
Received in revised form 29 May 2012
Accepted 30 May 2012
Available online 27 June 2012

Keywords:
Small regular graphs
Cage
Girth
Amalgam

ABSTRACT

In this paper we obtain $(q+3-u)$-regular graphs of girth 5, for $1 \leq u \leq q-1$ with fewer vertices than previously known ones, for each prime $q \geq 13$, performing operations of reductions and amalgams on the Levi graph B_q of an elliptic semiplane of type C. We also obtain a 13-regular graph of girth 5 on 236 vertices from B_{11} using the same technique.

1. Introduction

All graphs considered are finite, undirected and simple (without loops or multiple edges). For definitions and notations not explicitly stated the reader may refer to [11].

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *girth* of a graph G is the length $g = g(G)$ of a shortest cycle. The *degree* of a vertex $v \in V$ is the number of vertices adjacent to v. A graph is called *k-regular* if all its vertices have the same degree k, and *bi-regular* or (k_1, k_2)*-regular* if all its vertices have either degree k_1 or k_2. A (k, g)*-graph* is a k-regular graph of girth g and a (k, g)*-cage* is a (k, g)-graph with the smallest possible number of vertices. The necessary condition obtained from the distance partition with respect to a vertex yields a lower bound $n_0(k, g)$ on the number of vertices of a (k, g)-graph, known as the Moore bound.

$$n_0(k,g) = \begin{cases} 1 + k + k(k-1) + \cdots + k(k-1)^{(g-3)/2} & \text{if } g \text{ is odd;} \\ 2(1 + (k-1) + \cdots + (k-1)^{g/2-1}) & \text{if } g \text{ is even.} \end{cases}$$

Biggs [9] calls *excess* of a (k, g)-graph G the difference $|V(G)| - n_0(k, g)$. Cages have been intensely studied since they were introduced by Tutte [30] in 1947. Erdős and Sachs [15] proved the existence of a (k, g)-graph for any value of k and g. Since then, most of the work carried out has been focused on constructing smallest (k, g)-graphs (see e.g. [1,2,4–8,12,16,19,21,26,28,29,32]). Biggs is the author of a report on distinct methods for constructing cubic cages [10]. More details about constructions of cages can be found in the surveys by Wong [32], by Holton and Sheehan [23, Chapter 6], or the recent one by Exoo and Jajcay [18].

In this paper, for each prime $q \geq 13$, we construct a family of $(q+3-u)$-regular graphs of girth 5 which ties the order of $(q+3, 5)$-graphs from [24] for $u = 0$, and improves the known bounds for $1 \leq u \leq q-1$ (cf. Table 1).

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doi:10.1016/j.disc.2012.05.020

Table 1
Upper bounds for the order of $(k, 5)$-graphs.

k	Upper bound	Due to	New upper bound found in this paper
8	80	Royle	
9	96	Jørgensen	
10	126	Exoo	
11	156[a]	Jørgensen	
12	203	Exoo	
13	240[a]	Exoo	236
14	288	Jørgensen	284
15	312	Jørgensen	310
16	336	Jørgensen	[336]
17	448	Schwenk	
18	480	Schwenk	
19	512	Schwenk	
20	576	Jørgensen	572

General case: $(q + 3 - u, 5)$-graphs for $q \geq 13$ and $0 \leq u \leq q - 1$

q	u	Upper bound	Due to	New upper bound found in this paper
Prime	$u = 0$	$2(q^2 - 1)$	Jørgensen	$[2(q^2 - 1)]$
Prime	$1 \leq u \leq q - 1$	$2(q^2 - (q - 1)u - 1)$	Jørgensen	$2(q^2 - qu - 1)$
Prime power		$2(q^2 - (q - 1)u - 1)$	Jørgensen	

[a] Recently we have known that Exoo has constructed a (11, 5)-graph on 154 vertices and also a (13, 5)-graph on 230 vertices; see [17].

To construct such a family we use the following new technique that was inspired by papers from Funk [20] and Jørgensen [24]. We consider the Levi graph B_q of an elliptic semiplane of type C which is bipartite (cf. Section 2). Then we perform two reduction operations on the set of vertices of B_q (cf. Section 3) and an amalgam operation with bi-regular graphs (cf. Section 4) into the reduced graph. The novelty, with respect to [20,24], lies in performing Reduction 1 (cf. Section 3) before choosing the graphs for the amalgam.

Note that the general case presented in Section 5, holds for primes $q \geq 23$ (cf. Theorems 12 and 14). Smaller cases ($q = 13, 17, 19$) are treated with ad hoc similar constructions in Section 6, where we also obtain a 13-regular graph of girth 5 on 236 vertices from B_{11} which improves the bound found by Exoo in [17].

We conclude this section with Table 1 summarizing the state of the art regarding the new upper bounds for the order of $(k, 5)$-graphs from $k \geq 8$ (i.e. degrees k for which no cage has been constructed so far). The table is based on the values and references that appear in Table 4 of [18], and highlights the contributions of the results contained in this paper. The numbers in brackets indicate that the value found in this paper ties the previously known one.

2. Preliminaries

In this section we introduce the (bipartite) Levi graph B_q of an elliptic semiplane of type C [14,20] and we fix a labelling on its vertices which will be central for our construction since it allows us to keep track of the properties (such as regularity and girth) of the graphs obtained from B_q applying the reductions (cf. Section 3) and amalgams (cf. Section 4).

Definition 1. Let $GF(q)$ be a finite field with $q \geq 2$ a prime power. Let B_q be the Levi graph of an elliptic semiplane of type C which is a bipartite graph with vertex set (V_0, V_1) where $V_r = GF(q) \times GF(q)$, $r = 0, 1$, and edge set defined as follows:

$$(x, y)_0 \in V_0 \text{ is adjacent to } (m, b)_1 \in V_1 \text{ if and only if } y = mx + b. \quad (1)$$

This graph is also known as the incidence graph of the biaffine plane [22], and it has been used in the problem of finding extremal graphs without short cycles [13,25].

The following properties of the graph B_q are well known (see [22,25]) and they will be fundamental throughout the paper.

Proposition 2. *Let B_q be the (bipartite) Levi graph defined above. Let $P_x = \{(x, y)_0 | \ y \in GF(q)\}$, for $x \in GF(q)$ and $L_m = \{(m, b)_1 | \ b \in GF(q)\}$, for $m \in GF(q)$. Then the graph B_q has the following properties:*

(i) *it is q-regular, vertex transitive, of order $2q^2$, and has girth 6 for $q \geq 3$;*
(ii) *it admits a partition $V_0 = \bigcup_{x=0}^{q-1} P_x$ and $V_1 = \bigcup_{m=0}^{q-1} L_m$ of its vertex set;*
(iii) *each block P_x is connected to each block L_m by a perfect matching, for $x, m \in GF(q)$;*
(iv) *each vertex in P_0 and L_0 is connected* straight *to all its neighbours in B_q, meaning that $N((0, y)_0) = \{(i, y)_1 | i \in GF(q)\}$ and $N((0, b)_1) = \{(j, b)_0 | j \in GF(q)\}$;*
(v) *the other matchings between P_x and L_m are* twisted *and the rule is defined algebraically in $GF(q)$ according to (1);*
(vi) *it has diameter 4 and any two distinct points in P_x (or in L_m) are at distance exactly 4 for $x, m \in GF(q)$.*

For further information regarding these properties and for constructions of the adjacency matrix of B_q as a block (0, 1)-matrix please refer to [3,7].