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On the c-strong chromatic number of t-intersecting hypergraphs

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ABSTRACT

For a fixed $c \geq 2$, a *c-strong coloring* of the hypergraph G is a vertex coloring such that each edge e of G covers vertices with at least $\min\{c, |e|\}$ distinct colors. A hypergraph is *t-intersecting* if the intersection of any two of its edges contains at least t vertices. This paper addresses the question: what is the minimum number of colors which suffices to c-strong color any t-intersecting hypergraph? We first show that the number of colors required to c-strong color a hypergraph of size n is $O(\sqrt{n})$. Then we prove that we can use finitely many colors to 3-strong color any 2-intersecting hypergraph. Finally, we show that $2c-1$ colors are enough to c-strong color any shifted $(c-1)$-intersecting hypergraph, and $2c-2$ colors are enough to c-strong color any shifted t-intersecting hypergraph for $t \geq c$. Both chromatic numbers are optimal and match conjectured statements in which the shifted condition is removed.

1. Introduction

Graph and hypergraph coloring problems have a long history in combinatorics. Typically, two main types of vertex colorings of hypergraphs are considered. One is *weak coloring*, which is a vertex coloring such that none of the edges are monochromatic. Another type is *strong coloring*, which is a vertex coloring such that all the vertices in each edge have distinct colors. In the case of a graph, both notions coincide with the notion of a *proper* coloring, which is a vertex coloring such that the endpoints of each edge have distinct colors.

In their 2012 study of the property testing of boolean functions, Blais, Weinstein and Yoshida (BWY) [1] introduced a new kind of vertex coloring that generalizes both weak and strong colorings. They call this notion a *semi-strong* coloring.

Definition 1 ([1] *Semi-Strong Coloring, Chromatic Number*)**.** For a fixed $c \geq 2$, a *c-strong coloring* of the hypergraph G is an assignment of colors to its vertices such that each edge e of G contains vertices with at least $\min\{c, |e|\}$ distinct colors. The *c-strong chromatic number* of G, denoted by $\chi(G, c)$, is the minimum number of colors required to c-strong color G.

This paper focuses on the semi-strong coloring of intersecting hypergraphs. A hypergraph is *t-intersecting* if the intersection of any two of its edges contains at least t vertices.

Definition 2 ([1] *Chromatic Number of Intersecting Hypergraphs*)**.** Given two integers $t \geq 0$ and $c \geq 2$, the *c-strong chromatic number of t-intersecting hypergraphs*, denoted by $\chi(t, c)$, is the minimum number of colors that suffice to c-strong color any t-intersecting hypergraph. In other words,

$$\chi(t, c) = \max_{G \text{ is } t\text{-intersecting}} \chi(G, c).$$

If such a maximum does not exist, we write $\chi(t, c) = \infty$.

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If $\chi(t, c) = \infty$, it is of interest to know how the c-strong chromatic number grows as the size of the hypergraph increases.

Definition 3. Given integers $t \geq 0$, $c \geq 2$ and $n \geq 0$, define $\chi(t, c, n)$ to be the minimum number of colors that suffice to c-strong color any t-intersecting hypergraph of size at most n. That is,

$$\chi(t, c, n) = \max_{\substack{G \text{ is } t\text{-intersecting,} \\ |G| \leq n}} \chi(G, c).$$

Note that

$$\chi(t, c) = \lim_{n \to \infty} \chi(t, c, n).$$

We first restate the general bounds on $\chi(t, c)$ given in [1] (e is the base of the natural logarithm).

Proposition 1.1 (*[1]*). *For any $c \geq 2$,*

$$\begin{aligned} &\chi(t, c) = \infty \quad \text{for all } t \leq c - 2, \\ &\chi(c-1, c) \geq 2c - 1, \\ &\sqrt{c}e^c \geq \chi(t, c) \geq 2c - 2 \quad \text{for all } t \geq c, \\ &\chi(t, c) \leq 2c^2 \quad \text{for all } t \geq 2c. \end{aligned}$$

The authors of [1] further conjecture that both lower bounds are tight.

Conjecture 1.2 (*BWY*). *For any $c \geq 2$,*

$$\begin{aligned} &\chi(c-1, c) = 2c - 1, \\ &\chi(t, c) = 2c - 2 \quad \text{for all } t \geq c. \end{aligned}$$

We remark that $\chi(c-1, c)$ is not even known to be finite for any $c \geq 3$. Alon [2] has recently given better upper bounds on $\chi(t, c)$ for large t. In particular, Alon proved that $\chi(t, c)$ is linear in c for $t \geq 2c$, improving the previous result of BWY [1].

$$\begin{aligned} &\chi(t, c) \leq 2c - 1 \quad \text{for all } t \geq 2c^2, \\ &\chi(t, c) = O(c) \quad \text{for all } t \geq 2c. \end{aligned}$$

We adopt the following notations from [1] to facilitate the discussion:

$$[n] := \{1, 2, \ldots, n\}, \qquad \binom{[n]}{k} := \left\{ S \subset [n] \,\middle|\, |S| = k \right\}.$$

This paper is mainly concerned with providing evidence for the conjectures of BWY [1]. We first consider the boundary case $\chi(c-1, c)$ that is not even known to be finite, and show that the c-strong chromatic number grows somewhat slowly as the size of a $(c-1)$-intersecting hypergraph grows. We further show that it is indeed finite in the smallest unknown case, $\chi(2, 3)$. We also show that the conjectures are true for shifted hypergraphs.

We first give an upper bound on $\chi(c-1, c, n)$ asymptotically as $n \to \infty$.

Theorem 1.3. *For any fixed $c \geq 2$, we have*

$$\chi(c-1, c, n) = O(\sqrt{n}).$$

In the introductory paragraph of Section 2, we will remark that for any fixed $c \geq 2$, $\chi(t, c, n) = \Theta(n)$ for all $t \leq c - 2$ and $\chi(t, c, n) = \Theta(1)$ for all $t \geq c$, thus $t = c - 1$ is the only case in which the asymptotic behavior of $\chi(c-1, c, n)$ is interesting. The above result shows that even if $\chi(c-1, c)$ turns out to be infinite, $\chi(c-1, c, n)$ cannot grow very fast.

Next, we provide an upper bound on the value of $\chi(2, 3)$, thereby proving the conjecture of [1] that $\chi(c-1, c)$ is finite for the case where $c = 3$.

Theorem 1.4. $\chi(2, 3) \leq 21$.

Next, we prove that Conjecture 1.2 is true for *shifted* hypergraphs, which we define as follows.

If the vertices of a hypergraph $G = (V, E)$ are identified with the set $[n]$, where $n = |V|$, define a partial ordering on the edges by

$$\{a_1, \ldots, a_k\} \leq \{b_1, \ldots, b_k\}$$

if $a_i \leq b_i$ for all i, where $a = \{a_1 < \cdots < a_k\}$ and $b = \{b_1 < \cdots < b_k\} \in E$ are edges. A t-intersecting hypergraph $G = (V, E)$ with vertex set $[n]$ is said to be *shifted* if for all $e \in E$ and $e' \subset V$ with $e' \leq e$, we have $e' \in E$.

Shifted intersecting hypergraphs are the key objects of study in extremal set theory. They play a central role in the proofs of many important theorems in the field, for instance the Erdös–Ko–Rado theorem and Katona's theorem; see the survey by Frankl [3]. We prove Conjecture 1.2 for shifted intersecting hypergraphs.