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Note

On the characters of nilpotent association schemes

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ABSTRACT

In this paper we give some properties of the character values of nilpotent commutative association schemes. As a main result, a class of commutative Schurian association schemes is given.

1. Introduction

Nilpotent association schemes have been defined by Hanaki in [8] as a generalization of nilpotent finite groups. Some properties of the character values of nilpotent association schemes have also been given in [8]. In this paper, we first determine the character values of nilpotent commutative association schemes of class 2. Then we show that if (X, G) is a nilpotent commutative association scheme with the upper central series $\{1_X\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_{n-1} \subseteq G_n = G$ such that $\pi(G_i/G_{i-1})$, $1 \le i \le n$, are distinct sets, then for every faithful irreducible character $\chi \in \mathrm{Irr}(G/G_{i-1})$ and $g \in G - G_i$, where $1 \le i \le n-1$, we have $\chi(\sigma_g) = 0$. Moreover, we obtain a condition which a commutative association scheme is equal to the wreath product of some cyclic groups. More precisely, we show that if (X, G) is a nilpotent commutative association scheme of class n such that for every $1 \le i \le n$, G_i/G_{i-1} is a group of prime order p_i where p_i, $1 \le i \le n$, are distinct prime numbers, then we have $G = G_1 \wr (G_2/G_1) \wr (G_3/G_2) \wr \cdots \wr (G_n/G_{n-1})$.

2. Preliminaries

Let us first state some necessary definitions and notation. For details, we refer the reader to [1,12] for the background of association schemes. Throughout this paper, $\mathbb{C}$ denotes the complex numbers and $\mathbb{R}$ denotes the real numbers.

Definition 2.1. Let X be a finite set and G be a partition of $X \times X$. Then (X, G) is called an association scheme if the following properties hold:

(i) $1_X \in G$, where $1_X := \{(x, x) | x \in X\}$.
(ii) For every $g \in G$, g^* is also in G, where $g^* := \{(x, y) | (y, x) \in g\}$.
(iii) For every $g, h, k \in G$, there exists a nonnegative integer λ_{ghk} such that for every $(x, y) \in k$, there exist exactly λ_{ghk} elements $z \in X$ with $(x, z) \in g$ and $(z, y) \in h$.

For each $g \in G$, we call $n_g = \lambda_{gg^*1_X}$ the *valency* of g. For any nonempty subset H of G, put $n_H = \sum_{h \in H} n_h$. We call n_G the *order* of (X, G).

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Let H and K be nonempty subsets of G. We define HK to be the set of all elements $t \in G$ such that there exist element $h \in H$ and $k \in K$ with $\lambda_{hkt} \neq 0$. The set HK is called the *complex product* of H and K. If one of the factors in a complex product consists of a single element g, then one usually writes g for $\{g\}$. An association scheme (X, G) is called *commutative* if for all $g, h \in G, gh = hg$.

A nonempty subset H of G is called a *closed subset* if $HH \subseteq H$. A closed subset H of G is called *strongly normal* if $gHg^* = H$ for any $g \in G$.

Let H be a closed subset of G. For every $h \in H$ we define $xh = \{y \in X | (x, y) \in h\}$. Put $X/H = \{xH | x \in X\}$ and $G/H = \{g^H | g \in G\}$, where $xH = \bigcup_{h \in H} xh$ and $g^H = \{(xH, yH) | y \in xHgH\}$. Then $(X/H, G/H)$ is an association scheme called the *quotient scheme* of (X, G) over H. Note that a closed subset H is strongly normal iff the quotient scheme $(X/H, G/H)$ is a group with respect to the relational product iff $gg^* \subseteq H$, for every $g \in G$.

For each closed subset H of G, we define $O_\vartheta(H) = \{h \in H | n_h = 1\}$, called the *thin radical* of H. Note that $O_\vartheta(H)$ is a closed subset of G. In fact $O_\vartheta(H)$ is a group with respect to the relational product. The closed subset H is called *thin* if $O_\vartheta(H) = H$. A commutative association scheme (X, G) is called the *nilpotent* association scheme of class n, if it has the following sequence of closed subsets:

$$\{1_X\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_{n-1} \subseteq G_n = G$$

such that for all $1 \leq i \leq n$, $G_i/G_{i-1} = O_\vartheta(G/G_{i-1})$. We call this sequence the *upper central series* of (X, G).

Let (X, G) be an association scheme. For every $g \in G$, let σ_g be the adjacency matrix of g. That is, σ_g is the matrix whose rows and columns are indexed by the elements of X and its (x, y)-entry is 1 if $(x, y) \in g$ and 0 otherwise. Note that for every $h, k \in G$, $\sigma_h \sigma_k = \sum_{t \in G} \lambda_{hkt} \sigma_t$. For any nonempty subset H of G, we put $\sigma_H := \{\sigma_h | h \in H\}$.

It is known that $\mathbb{C}G = \bigoplus_{g \in G} \mathbb{C}\sigma_g$, the *adjacency algebra* of (X, G), is a semisimple algebra. The set of irreducible characters of G is denoted by $\mathrm{Irr}(G)$.

Let (X, G) be an association scheme and $\chi, \psi \in \mathrm{Irr}(G)$. In [5], the character product of χ and ψ is defined by $\chi\psi(\sigma_g) = \frac{1}{n_g}\chi(\sigma_g)\psi(\sigma_g)$. It is known that this character product need not be a character in general.

In the following, we deal with the dual of the complex adjacency algebra of a commutative association scheme in the sense of [1]. We refer the reader to [2] for the background of table algebras.

Suppose that $(\mathbb{C}G, \sigma_G)$ is the complex adjacency algebra of a commutative association scheme (X, G), where $\sigma_G = \{\sigma_g | \, g \in G\}$. Let $\{\varepsilon_\chi | \, \chi \in \mathrm{Irr}(G)\}$ be the set of the primitive idempotents of $\mathbb{C}G$. Then from [1, Section 2.5] there are two matrices $P = (p_g(\chi))$ and $Q = (q_\chi(g))$ in $\mathrm{Mat}_{|X|}(\mathbb{C})$, where $g \in G$ and $\chi \in \mathrm{Irr}(G)$, such that $PQ = QP = |X|I$, where I is the identity matrix in $\mathrm{Mat}_{|X|}(\mathbb{C})$, and

$$\sigma_g = \sum_{\chi \in \mathrm{Irr}(G)} p_g(\chi)\varepsilon_\chi \quad \text{and} \quad \varepsilon_\chi = \frac{1}{|X|}\sum_{g \in G} q_\chi(g)\sigma_g.$$

The dual of $(\mathbb{C}G, \sigma_G)$ in the sense of [1] is as follows: with each linear representation $\Delta_\chi : \sigma_g \mapsto p_g(\chi)$, we associate the linear mapping $\Delta^*_\chi : \sigma_g \mapsto q_\chi(g) = \frac{m_\chi \chi(\sigma_{g^*})}{n_g}$. Since the matrix $Q = (q_\chi(g))$ is non-singular, the set $B = \{\Delta^*_\chi : \chi \in \mathrm{Irr}(G)\}$ is linearly independent and so forms a base of the set of all linear mappings A of $\mathbb{C}G$ into $\mathbb{C}$. From [1, Theorems 5.9 and 3.8] the pair (A, B) is a table algebra (see [2]) with the identity $1_A = \Delta^*_\rho$, where $\rho \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}G, \mathbb{C})$ such that $\rho(\sigma_g) = n_g$, and involutory automorphism $*$ which maps Δ^*_χ to $\Delta^*_{\overline{\chi}}$, where $\overline{\chi}$ is the complex conjugate to χ. The table algebra (A, B) is called the *dual* of the association scheme (X, G). Moreover, for every $\chi, \psi \in \mathrm{Irr}(G)$ we have

$$\Delta^*_\chi \Delta^*_\psi = \sum_{\Delta^*_\varphi \in B} q^\varphi_{\chi\psi} \Delta^*_\varphi$$

where the structure constants $q^\varphi_{\chi\psi}$, $\varphi \in \mathrm{Irr}(G)$ are nonnegative real numbers. For every $\chi, \psi \in \mathrm{Irr}(G)$, put

$$\mathrm{Supp}(\Delta^*_\chi \Delta^*_\psi) = \{\Delta^*_\varphi \in B | q^\varphi_{\chi\psi} \neq 0\}.$$

A nonempty subset N of B is called a *closed subset* if for every $\Delta^*_\chi, \Delta^*_\psi \in N$ we have $\mathrm{Supp}(\Delta^*_\chi \Delta^*_\psi) \subseteq N$.

Let (A, B) be the dual of $(\mathbb{C}G, \sigma_G)$ and let H be a closed subset of G. Put

$$\mathrm{Ker}(H) = \{\Delta^*_\chi \in B \mid \chi(\sigma_g) = n_g, \text{ for every } g \in H\}.$$

Then from [2, Theorems 1 and 2], $\mathrm{Ker}(H)$ is a closed subset of B and

$$\widehat{H} \simeq B/\mathrm{Ker}(H)$$

where $\widehat{H}$ is the dual of $\mathbb{C}H$.

3. Main results

In this section we assume that (X, G) is a commutative association scheme. We first give some properties of the character values of nilpotent association schemes of class 2.